

# SPECTRAL THEORY FOR GAUSSIAN PROCESSES: REPRODUCING KERNELS, RANDOM FUNCTIONS, BOUNDARIES, AND $L^2$ -WAVELET GENERATORS WITH FRACTIONAL SCALES

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*To the Memory of William B. Arveson*

**ABSTRACT.** A recurrent theme in functional analysis is the interplay between the theory of positive definite functions, and their reproducing kernels, on the one hand, and Gaussian stochastic processes, on the other. This central theme is motivated by a host of applications, e.g., in mathematical physics, and in stochastic differential equations, and their use in financial models. In this paper, we show that, for three classes of cases in the correspondence, it is possible to obtain explicit formulas which are amenable to computations of the respective Gaussian stochastic processes. For achieving this, we first develop two functional analytic tools. They are: (i) an identification of a universal sample space  $\Omega$  where we may realize the particular Gaussian processes in the correspondence; and (ii) a procedure for discretizing computations in  $\Omega$ . The three classes of processes we study are as follows: Processes associated with: (a) arbitrarily given sigma finite regular measures on a fixed Borel measure space; (b) with Hilbert spaces of sigma-functions; and (c) with systems of self-similar measures arising in the theory of iterated function systems. Even our results in (a) go beyond what has been obtained previously, in that earlier studies have focused on more narrow classes of measures, typically Borel measures on  $\mathbb{R}^n$ . In our last theorem (section 10), starting with a non-degenerate positive definite function  $K$  on some fixed set  $T$ , we show that there is a choice of a universal sample space  $\Omega$ , which can be realized as a boundary of  $(T, K)$ . Its boundary-theoretic properties are analyzed, and we point out their relevance to the study of electrical networks on countable infinite graphs.

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## 1. INTRODUCTION

We are considering three functional analytic questions arising at the crossroads of pure and applied probability theory. In different contexts of non-deterministic analysis, one needs mathematical representations of the set of all possible outcomes, called *the sample space*  $\Omega$ , of some experiment, for example involving random trials. This is easy enough in simple discrete models, for example in experiment with tossing coins. The sample space of each trial is the set  $\{\text{head}, \text{tail}\}$ , and more subtle models then involve Cartesian products. However in infinite models, and in most continuous models, a complete description of a sample space of outcomes and its subsets, events, presents subtle problems. In Brownian motion models, for example,  $\Omega$  may conveniently be taken to be a suitable space of continuous functions, sample paths. Now, to approach computations, one is faced with the use of simulations of suitable subsets in  $\Omega$ ; e.g., Monte-Carlo simulations. For such approaches, because of noise, of uncertainties, or limited information, it is often helpful to pick different mathematical realizations of the set  $\Omega$ : For example, a version of  $\Omega$  consisting of sample paths defined only on suitable subsets, as opposed to defined point-wise. This is often good enough as one is interested in particular functions on  $\Omega$ . Whichever choice is made,  $\Omega$  will naturally come equipped with a sigma-algebra, say  $\mathcal{F}$ , of subsets, and a probability measure  $P$  defined on  $\mathcal{F}$ . The  $\mathcal{F}$ -measurable functions are random variables, and systems of random variables are stochastic processes. The process is Gaussian if we can

choose the probability measure  $P$  such that the random variables making up the process are Gaussian, and in  $\mathbf{L}^2(P)$ .

With the use of the corresponding Gaussian densities, and covariance functions, one then computes quantities from the random variables; and the question of choice of  $\Omega$  can then often be avoided. Nonetheless, for applications to stochastic integration, one is forced to be more precise with the choice of  $\Omega$ , and a number of functional analytic tools are available for the purpose. In the approach to this problem based on Gelfand-triples (see Section 3), one may realize  $\Omega$  as a space of Schwartz-tempered distributions. However with this realization of  $\Omega$ , it is more difficult to make a direct connection to the initial model, and to set up suitable Monte-Carlo simulations. As a result, there is a need for discretizations. Several such discretizations will be presented here, and comparisons will be made.

Our approach, in this general context, relies on our use of Gaussian Hilbert spaces, and of associated sequences of independent, identically distributed (i. i. d.) standard Gaussian  $N(0, 1)$ -random variables. But this then further introduces a host of choices, and of these we identify one which is universal in a sense made precise in Sections 3-7.

In this paper, we will focus on Gaussian stochastic processes, but we also offer applications of our results to certain random functions (Section 9) which involve non-Gaussian distributions. Similarly, a host of simulation approaches involve non-Gaussian choices.

The purpose of the paper is three-fold. First we study (i) a universal choice of sample space for a family of  $\mathbf{L}^2$  Gaussian noise processes. While these processes have appeared in one form or the other in prior literature, the choice of sample spaces has not been studied in a way that facilitates comparisons. We index these Gaussian noise processes by the set of regular measures in some fixed measure space  $(M, \mathcal{B})$ , with  $\mathcal{B}$  some given Borel sigma-algebra of subsets in  $M$ . Secondly we make precise (ii) equivalence in this category of Gaussian noise processes, and we prove a uniqueness theorem, where uniqueness is specified by a specific measure isomorphism of the respective sample spaces. In our third result (iii), given a fixed measure space  $(M, \mathcal{B})$ , we identify a Hilbert space  $\mathcal{H}$ , with the property that the Gaussian noise process indexed by  $\mathcal{H}$  is universal envelope of all the Gaussian noise processes from (ii). As applications we compute Gaussian noise processes associated to Cantor measures, and more generally to iterated function

systems (IFS) measures, and to a family of reproducing kernel Hilbert spaces (RKHS).

For readers not familiar with Gaussian processes, for the present purpose, the following are helpful: [3, 6, 9, 19, 22, 23, 24, 45, 46]; for infinite products and applications, see [34], [49], and [4], [7]. The universal Hilbert space from (iii) is used in a different context [11, 22, 28, 37, 38]. For a small sample of recent applications, we cite [16, 21, 27]. For reproducing kernel Hilbert spaces, see, for example, [2, 44]. In the way of presentation, it will be convenient to begin with a quick review of infinite products, this much inspired by the pioneering paper [34] by Kakutani.

## 2. PRELIMINARIES

Below we present a framework of Gaussian Hilbert spaces. These in turn play a crucial role in the study of positive semi-definite kernels, and their associated reproducing kernel Hilbert spaces, see Sections 9-10. In its most general form, the theory of Gaussian Hilbert spaces  $\mathcal{H}$  is somewhat abstract, and it is therefore of interest, for particular cases of  $\mathcal{H}$ , to study natural decompositions into cyclic components in  $\mathcal{H}$  which arise in applications, and admit computation. Hence we begin with those processes whose covariance function may be determined by a fixed measure. Even this simpler case generalizes a host of Gaussian processes studied earlier with the use of Gelfand triples built over the standard Hilbert space  $\mathbf{L}^2(\mathbb{R}^d, dx)$ , with  $dx$  denoting the Lebesgue measure, with the use of Laurent Schwartz theory of tempered distributions. Our present framework is not confined to the Euclidean case. Indeed, starting with any measure space  $M$  and a Borel sigma-algebra  $\mathcal{B}$ , we then show in Section 5 that the General Gaussian Hilbert space (Definition 2.2) decomposes as an orthogonal sum where the corresponding cyclic subspaces are those generated by a family of sigma-finite measures on  $M$ . Indeed, in applications to measurement, in physics, and in statistics, it is often not possible to pin down a variable as a function of points in the underlying space  $M$ . As a result, it has proved useful to study processes indexed by sigma-algebras of subsets of  $M$ .

In our consideration of random variables, of Hilbert spaces, and of Gaussian stochastic processes, it will be convenient for us to restrict to the case of *real-valued* functions and real Hilbert spaces. It will be helpful to first state the respective results in the real case, and

then, at the end, when needed, remove the restriction. One instance when complex Hilbert spaces are needed is the introduction of Fourier bases, i.e., orthogonal bases consisting of functions  $e_\lambda$  where  $\lambda \in \mathbb{R}$  and  $e_\lambda(x) = e^{i\lambda x}$  or  $e^{2\pi i\lambda x}$ . However, our setting will be general measure spaces  $(M, \mathcal{B}, \mu)$ , where  $\mathcal{B}$  is a sigma-algebra of measurable subsets of some set  $M$ , and  $\mu$  is a positive measure on  $M$ . The restricting assumption is *sigma-finiteness*, i.e., there are subsets  $B_1, B_2, \dots$  of  $\mathcal{B}$  such that

$$(2.1) \quad M = \bigcup_{j=1}^{\infty} B_j, \quad \text{and} \quad \mu(B_j) < \infty, \quad \forall j \in \mathbb{N}.$$

**Definition 2.1.** A Gaussian (noise) stochastic process indexed by  $(M, \mathcal{B}, \mu)$  consists of a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ :  $\Omega$  is a set (sample space),  $\mathcal{F}$  is a sigma-algebra of subsets (events) of  $\Omega$ , and  $\mathbb{P}$  is a probability measure defined on  $\mathcal{F}$ . We assume that, for all  $A \in \mathcal{B}$  such that  $\mu(A) < \infty$ , there is a Gaussian random variable

$$(2.2) \quad W_A = W_A^{(\mu)} : \Omega \longrightarrow \mathbb{R}$$

with zero mean and variance  $\mu(A)$  (that is,  $W_A \sim N(0, \mu(A))$ ), the Gaussian with zero mean and variance  $\mu(A)$ ), i.e. for all  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ ,

$$\{\omega \in \Omega \mid a < W_A(\omega) \leq b\} \in \mathcal{F},$$

and

$$\begin{aligned} \mathbb{P}(\{a < W_A(\omega) \leq b\}) &= \int_a^b \frac{1}{\sqrt{2\pi\mu(A)}} e^{-\frac{x^2}{2\mu(A)}} dx \\ &= \int_{\frac{a}{\sqrt{\mu(A)}}}^{\frac{b}{\sqrt{\mu(A)}}} \frac{1}{\sqrt{2\pi\mu(A)}} e^{-\frac{x^2}{2}} dx \\ &= \gamma_1\left(\left(\frac{a}{\sqrt{\mu(A)}}, \frac{b}{\sqrt{\mu(A)}}\right]\right), \end{aligned}$$

where  $\gamma_1$  is the standard Gaussian on  $\mathbb{R}$ .

**Definition 2.2.** A Gaussian process indexed by a (fixed) Hilbert space  $\mathcal{H}$  consists of a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that, for all  $F \in \mathcal{H}$ , there is a Gaussian random variable  $W_F$  with law  $N(0, \|F\|_{\mathcal{H}}^2)$  such that

$$(2.3) \quad \mathbb{E}(W_{F_1} W_{F_2}) = \langle F_1, F_2 \rangle_{\mathcal{H}}, \quad \forall F_1, F_2 \in \mathcal{H}.$$

It is further assumed that for all  $\{A_j\}_{j=1}^n \subset \mathcal{B}$  such that  $0 < \mu(A_j) < \infty$ ,  $i = 1, 2, \dots, n$ , the joint distribution of the family  $\{W_{A_j}\}_{j=1}^n$  is Gaussian with zero mean and covariance matrix  $(\mu(A_i \cap A_j))_{i,j=1}^n$ . We

will assume throughout that  $\dim \mathbf{L}^2(\mu) = \infty$ . The finite dimensional case is dealt with separately.

**Remark 2.3.** *With the specifications in Definitions 2.1 and 2.2, it is known that, in each case, such Gaussian processes exist. In the case of Definition 2.2, when  $\{F_i\}_{i=1}^n$  is a system in  $\mathcal{H}$ , then the random variables  $\{W_{F_i}\}_{i=1}^n$  have a joint Gaussian distribution corresponding to the covariance matrix  $(\langle F_i, F_j \rangle_{\mathcal{H}})_{i,j=1}^n$ .*

Starting with a measure space  $(M, \mathcal{B})$ , we will show in Section 5, that there is a universal Hilbert space  $\mathcal{H}$  which contains all the stochastic processes derived from sigma-finite measures  $\mu$  on  $(M, \mathcal{B})$ . In detail, given an arbitrary  $\mu$ , we get a Gaussian process  $W^{(\mu)}$  with  $\mu$  as its covariance measure; see Definition 2.1. Now, the universal Hilbert space  $\mathcal{H}$  over  $(M, \mathcal{B})$  will satisfy the conditions in Definition 2.2; and it will be a Hilbert space of sigma-functions (Definition 4.1). Before getting to this, we must prepare the ground with some technical tools. This is the purpose of the next section on infinite products, and discrete Gelfand-triples.

### 3. THE PROBABILITY SPACE $(\Omega_{\mathbf{s}}, \mathcal{F}_{\mathbf{s}}, Q)$

The purpose of this section is to show that there is a single infinite-product measure space such that for every measure space  $M$  and fixed Borel sigma-algebra  $\mathcal{B}$ , everyone of the Gaussian processes  $W^{(\mu)}$ , where  $\mu$  is sigma-finite measure on  $M$ , may be represented in  $\mathbf{L}^2$  of this infinite-product measure space. Since the construction must apply to every sigma-finite measure  $\mu$ , we must adjust the construction so that it can be adapted to orthonormal bases (ONBs) in each of the corresponding  $\mathbf{L}^2(\mu)$  Hilbert spaces. To do this, we will be introducing a suitable Gelfand triple (see (3.2)-(3.3)), realized in sequence spaces, as opposed to the more traditional setting based instead on  $\mathbf{L}^2(R^d, dx)$  and Schwartz tempered distributions. There is a number of advantages of this approach, for example we are not singling out any particular  $\mathbf{L}^2(\mu)$ , and also not a particular choice of ONB.

An initial choice for  $\Omega_S$  is  $\Omega_S = \times_{\mathbb{N}} \mathbb{R}$ , that is the space of all functions from  $\mathbb{N}$  into  $\mathbb{R}$ , or equivalently, of all real sequences  $(c_1, c_2, \dots)$  indexed by  $\mathbb{N}$ . Let  $\mathbf{s}$  be the space of sequences  $c = (c_n)_{n \in \mathbb{N}} \in \times_{\mathbb{N}} \mathbb{R}$  with the following property: For every  $p \in \mathbb{N}$  there exists  $K_p < \infty$  such that

$$(3.1) \quad |c_j| \leq K_p j^{-p}, \quad \forall j \in \mathbb{N},$$

and denote by  $\mathbf{s}'$  the dual space of all sequences  $\xi = (\xi_j)_{j \in \mathbb{N}}$  of polynomial growth, that is, such that there exists  $q \in \mathbb{N}$  and  $K_q > 0$  such

that

$$(3.2) \quad |\xi_j| \leq K_q j^q, \quad \forall j \in \mathbb{N}.$$

Then (see [23, 25, 43])

$$\mathbf{s} \subset \ell^2 \subset \mathbf{s}'$$

is a Gelfand triple, i.e., with the semi-norms defined from (3.1),  $\mathbf{s}$  becomes a Fréchet space, and the embedding from  $\mathbf{s}$  into  $\ell^2$  is nuclear (and  $\mathbf{s}'$  denotes the dual of  $\mathbf{s}$ ).

Let  $\mathcal{F}_{\mathbf{s}}$  denote the sigma-algebra of subsets in  $\mathbf{s}'$  generated by the cylinder sets as follows: For  $c_1, c_2, \dots, c_n \in \mathbf{s}$  and an open set  $O \subset \mathbb{R}^n$ , define the cylinder  $\text{Cyl}(c_1, \dots, c_n, O)$  by

$$(3.3) \quad \text{Cyl}(c_1, \dots, c_n, O) = \{\xi \in \mathbf{s}' \mid (\langle \xi, c_1 \rangle, \dots, \langle \xi, c_n \rangle) \in O\}.$$

As the data in (3.3) varies, we get the cylinder sets in  $\mathbf{s}'$  and the corresponding sigma-algebra  $\mathcal{F}_{\mathbf{s}}$ .

Further note that the sets in (3.3) generate a system of neighborhoods for the weak\*-topology on  $\mathbf{s}'$ . Moreover, if  $\mathbf{s}$  is assigned its Fréchet topology from the semi-norms in (3.1), then  $\mathbf{s}'$  (with its weak\*-topology) is the dual of  $\mathbf{s}$ .

**Lemma 3.1.** *From Gelfand's theory we therefore get the existence of a unique probability measure  $Q$  on  $(\mathbf{s}', \mathcal{F}_{\mathbf{s}})$  with the property that*

$$(3.4) \quad \int_{\mathbf{s}'} e^{i\langle \xi, c \rangle} dQ(\xi) = e^{-\frac{1}{2}\|c\|_2^2},$$

where

$$(3.5) \quad \langle \xi, c \rangle = \sum_{j=1}^{\infty} c_j \xi_j,$$

and  $\|c\|_2^2 = \sum_{j=1}^{\infty} c_j^2$ . Moreover,  $\langle \xi, c \rangle$  in (3.4) and (3.5) extends from  $\mathbf{s} \times \mathbf{s}'$  to  $\ell^2 \times \mathbf{s}'$ , representing every  $c \in \ell^2$  as a Gaussian variable on  $(\mathbf{s}', \mathcal{F}_{\mathbf{s}}, Q)$ , with

$$(3.6) \quad \mathbf{E}_Q(\langle \cdot, c \rangle) = 0,$$

and

$$(3.7) \quad \mathbf{E}_Q(\langle \cdot, c \rangle^2) = \|c\|_2^2.$$

Furthermore, the set of coordinate functions on  $\Omega_{\mathbf{s}} : \mathbf{s}'$ ,

$$\pi_j(\xi) = \xi_j, \quad j \in \mathbb{N},$$

turns into an independent, identically distributed (i.i.d.) system of  $N(0, 1)$  standard Gaussian variables, and we get:

$$(3.8) \quad \mathbf{E} \left( \pi_{j_1} \pi_{j_2} \cdots \pi_{j_k} e^{i\langle \cdot, c \rangle} \right) = (-1)^{k/2} c_{j_1} c_{j_2} \cdots c_{j_k} e^{-\frac{1}{2} \|c\|_2^2}.$$

**Proof:** We begin with the assertion

$$(3.9) \quad \mathbb{E}_Q (\pi_j \pi_k) = \delta_{j,k}, \quad \forall j, k \in \mathbb{N}.$$

Take first  $j = k$ ; then,

$$\mathbb{E}_Q (\pi_j^2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} c_j^2 e^{-\frac{c_j^2}{2}} dc_j = 1,$$

and if  $j \neq k$  we get

$$\begin{aligned} \mathbb{E}_Q (\pi_j \pi_k) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} c_j c_k e^{-\frac{c_j^2 + c_k^2}{2}} dc_j dc_k \\ &= \left( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{x^2}{2}} dx \right)^2 \\ &= 0, \end{aligned}$$

which proves (3.9).

We now prove the assertion (3.4). With (3.5) we get

$$\langle \xi, c \rangle = \sum_{j=1}^{\infty} c_j \pi_j(\xi), \quad \forall \xi \in \mathbf{s}', \text{ and } \forall c \in \mathbf{s}.$$

We prove (3.4) for  $c \in \mathbf{s}$ , and then extend it to all of  $\ell^2$ . We have

$$\begin{aligned} \int_{\mathbf{s}'} e^{i\langle \xi, c \rangle} dQ(\xi) &= \mathbb{E}_Q \left( e^{i \sum_{k=1}^{\infty} c_k \pi_k(\cdot)} \right) \\ &= \prod_{k=1}^{\infty} \mathbb{E}_Q \left( e^{i c_k \pi_k(\cdot)} \right) \\ &= \prod_{k=1}^{\infty} e^{-\frac{c_k^2}{2}} \\ &= e^{-\frac{1}{2} \|c\|_2^2}, \end{aligned}$$

which is the desired conclusion.

The proof of (3.8) follows from an application of (3.4) to

$$(3.10) \quad c + t_1 e_{j_1} + \cdots + t_k e_{j_k}, \quad t_1, \dots, t_k \in \mathbb{R},$$

where

$$(e_j)_\ell := \delta_{j,\ell}, \quad \forall j, \ell \in \mathbb{N}.$$



is the standard ONB in  $\ell^2$ , i.e.  $\tilde{e}_j = \pi_j$ . Now (3.8) follows if (3.10) is substituted into (3.4), and the partial derivatives  $\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k}$  are computed on both sides, and then evaluated at  $t_1 = \cdots = t_k = 0$ .  $\square$

**Lemma 3.2.** *Let  $Q = \times_{\mathbb{N}} \gamma_1$  be the product measure. Then*

$$(3.11) \quad Q(\mathbf{s}') = 1 \quad \text{and} \quad Q(\ell^2) = 0.$$

**Proof:** The first claim follows from Minlos's theorem applied to the following positive definite function on  $\mathbf{s}$

$$(3.12) \quad c \in \mathbf{s} \quad \mapsto \quad e^{-\frac{\|c\|^2}{2}}.$$

Indeed, the function (3.12) is clearly continuous with respect to the semi-norms in  $\mathbf{s}$ ; see (3.1).

We need to prove the second claim in (3.11), i.e. the assertion that  $\ell^2 \subset \mathbf{s}'$  has  $Q$ -measure zero. Assume the contrary, i.e. assume  $Q(\ell^2) > 0$ . Since

$$\lim_{j \rightarrow \infty} \pi_j = 0$$

point-wise on  $\ell^2$ , we have

$$\lim_{j \rightarrow \infty} \int_{\ell^2} e^{i\pi_j(\omega)} dQ(\omega) = Q(\ell^2)$$

as an application of Lebesgue's dominated convergence theorem. On the other hand,

$$\mathbb{E}_Q(e^{i\pi_k(\cdot)}) = e^{-\frac{1}{2}}, \quad \forall k \in \mathbb{N},$$

and so another application of Lebesgue's dominated convergence theorem leads to

$$e^{-\frac{1}{2}} = Q(\ell^2) + Q(\mathbf{s}' \setminus \ell^2),$$

which is a contradiction since the sum should be equal to 1.  $\square$

**Theorem 3.3.** *Let  $(M, \mathcal{B}, \mu)$  be a sigma-finite measure space as specified in Section 4, and let  $\{\varphi_j\}_{j \in \mathbb{N}}$  be a choice of orthonormal basis (ONB) in  $\mathbf{L}^2(\mu)$ . Then, the Gaussian process  $W^{(\mu)}$  may be realized in  $\mathbf{L}^2(\Omega_{\mathbb{S}}, \mathcal{F}_{\mathbf{s}}, Q)$  as follows: For  $A \in \mathcal{B}$  such that  $0 < \mu(A) < \infty$ , set*

$$(3.13) \quad W_A^{(\mu)}(\xi) = \sum_{j=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right) \pi_j(\xi), \quad \xi \in \mathbf{s}'.$$

*Then,  $W^{(\mu)}$ , defined by (3.13), is a copy of the Gaussian process from Definition 2.1.*

**Proof:** In view of (3.8), we need only to prove that  $W_A^{(\mu)}$  in (3.13) is a  $N(0, \mu(A))$  Gaussian variable, and that

$$(3.14) \quad \mathbb{E}_Q \left( W_A^{(\mu)} W_B^{(\mu)} \right) = \mu(A \cap B), \quad \forall A, B \in \mathcal{B}.$$

But the first assertion follows from

$$\sum_{j=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right)^2 = \|\chi_A\|_{\mathbf{L}^2(\mu)}^2 = \mu(A),$$

and we prove (3.14) as follows:

$$\begin{aligned} \mathbb{E}_Q \left( \left( \sum_{j=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right) \pi_j \right) \left( \sum_{k=1}^{\infty} \left( \int_B \varphi_k(x) d\mu(x) \right) \pi_k \right) \right) &= \\ &= \sum_{j,k=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right) \left( \int_B \varphi_k(x) d\mu(x) \right) \delta_{j,k} \\ &= \langle \chi_A, \chi_B \rangle_{\mathbf{L}^2(\mu)} \\ &= \mu(A \cap B), \end{aligned}$$

which is the desired conclusion.  $\square$

In the next section, we generalize the expansion formula (3.13) above.

**Corollary 3.4.** *Let  $(M, \mathcal{B})$  be as in Theorem 3.3, let  $\mu$  and  $\lambda$  be two sigma-finite measures defined on it, such that  $\mu \ll \lambda$ , and let  $f \in \mathbf{L}^2(M, \mathcal{M}, \mu)$ . Then, in the representation (3.13), referring to  $\mathbf{L}^2(\mathbf{s}', Q)$ , we have*

$$(3.15) \quad W^{(\lambda)} \left( f \sqrt{\frac{d\mu}{d\lambda}} \right) = W^{(\mu)}(f), \quad Q \text{ a.e.}$$

**Proof:** Picking an ONB  $\{\varphi_j\}_{j \in \mathbb{N}}$  in  $\mathbf{L}^2(M, \mathcal{M}, \mu)$ , we note that then  $\left\{ \varphi_j \sqrt{\frac{d\mu}{d\lambda}} \right\}_{j \in \mathbb{N}}$  is an ONB in  $\mathbf{L}^2(M, \mathcal{M}, \lambda)$ . Now use (3.13) for the pair

of ONBs. We get

$$\begin{aligned}
W^{(\mu)}(f) &= \sum_{j=1}^{\infty} \langle \varphi_j, f \rangle_{\mathbf{L}^2(\mu)} \pi_j \\
&= \sum_{j=1}^{\infty} \left( \int_M \varphi_j(x) f(x) d\mu(x) \right) \pi_j \\
&= \sum_{j=1}^{\infty} \left( \int_M \varphi_j(x) f(x) \frac{d\mu}{d\lambda}(x) d\lambda(x) \right) \pi_j \\
&= \sum_{j=1}^{\infty} \left( \int_M \sqrt{\frac{d\mu}{d\lambda}}(x) \varphi_j(x) \sqrt{\frac{d\mu}{d\lambda}}(x) f(x) d\lambda(x) \right) \pi_j \\
&= W^{(\lambda)} \left( f \sqrt{\frac{d\mu}{d\lambda}} \right).
\end{aligned}$$

□

We need another construction of a universal space as well, using a construction of Kakutani [34]. More precisely, consider the space  $\times_{\mathbb{N}} \mathbb{R}$ , and denote by  $\xi$  a running element in this cartesian product. Define for  $F(\xi) = f_n(\xi_1, \dots, \xi_n)$ , where  $f_n$  is a measurable and summable function of  $n$  real variables

$$\mathcal{L}(F) = \int \cdots \int_{\mathbb{R}^n} f_n(\xi_1, \dots, \xi_n) \gamma_n(\xi_1, \dots, \xi_n) d\xi_1 \cdots d\xi_n,$$

where  $\gamma_n$  is the product of the densities of  $n$  i.i.d.  $N(0, 1)$  variables. By Kolmogorov's theorem [40], there exists a unique probability  $Q_K$  on  $\prod_{\mathbb{N}} \mathbb{R}$  such that

$$\int_{\prod_{\mathbb{N}} \mathbb{R}} F(\xi) dQ_K(\xi) = \int \cdots \int_{\mathbb{R}^n} f_n(\xi_1, \dots, \xi_n) \gamma_n(\xi_1, \dots, \xi_n) d\xi_1 \cdots d\xi_n.$$

In fact,

$$(3.16) \quad Q_K = \prod_{\mathbb{N}} \gamma_1$$

on the countably infinite Cartesian product  $\prod_{\mathbb{N}} \mathbb{R}$ , where  $\gamma_1$  is the standard  $N(0, 1)$  Gaussian on  $\mathbb{R}$  (with density  $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ ).

The measure  $Q_K$  and  $Q$  have the same characteristic function, and  $Q_K(\mathbf{s}') = 1$ . So we will in the sequel use both the spaces  $(\mathbf{s}', \mathcal{F}_{\mathbf{s}}, Q)$  and  $(\prod_{\mathbb{N}} \mathbb{R}, Q_K)$ .

#### 4. A HILBERT SPACE OF SIGMA-FUNCTIONS

In spectral theory, in representation theory (see e.g. [11, 38]), and in the study of infinite products [22], and of iterated function systems (IFS) (see e.g. [26]) one is faced with the problem of identifying direct integral decompositions. Naturally, a given practical problem may not by itself entail a Hilbert space, and, as a result, one must be built by use of the inherent geometric features of the problem. In these applications it has proved useful to build the Hilbert space from a set of equivalence classes. The starting point will be pairs  $(f, \mu)$  where  $\mu$  is a measure, and  $f$  is a function, assumed in  $\mathbf{L}^2(\mu)$ . It turns out (see [38]) that the set of such equivalence classes acquire the structure of a Hilbert space, called a sigma-Hilbert space. Further we show through applications (Sections 7 and 8) that these sigma-Hilbert spaces form a versatile tool in the study of Gaussian processes. These Gaussian processes are indexed by a choice of a suitable sigma-algebras of subsets of  $M$ .

**Definition 4.1.** *Let  $(M, \mathcal{B})$  be a fixed measure space, and let  $\mathcal{M}(M, \mathcal{B})$  denote the set of all sigma-finite positive measures on  $(M, \mathcal{B})$ . For pairs  $(f_i, \mu_i)$ ,  $i = 1, 2$ , where*

$$(4.1) \quad \mu_i \in \mathcal{M}(M, \mathcal{B}), \quad f_i \in \mathbf{L}^2(\mu_i),$$

*we introduce the equivalence relation  $\sim$  as follows:  $(f_1, \mu_1) \sim (f_2, \mu_2)$  if and only if there exists  $\lambda \in \mathcal{M}(M, \mathcal{B})$  such that  $\mu_i \ll \lambda$  and*

$$(4.2) \quad f_1 \sqrt{\frac{d\mu_1}{d\lambda}} = f_2 \sqrt{\frac{d\mu_2}{d\lambda}}, \quad \lambda \text{ a.e.}$$

*Here,  $\frac{d\mu_i}{d\lambda}$  denote the respective Radon-Nikodym derivatives. For the measure  $\lambda$ , we may take  $\lambda = \mu_1 + \mu_2$ .*

It is known (see [38]), that (4.2) indeed defines an equivalence relation in the set of all pairs as specified in (4.1). If  $\mu \in \mathcal{M}(M, \mathcal{B})$  and  $f \in \mathbf{L}^2(\mu)$ , we denote the equivalence class of  $(f, \mu)$  by  $f\sqrt{d\mu}$ . Moreover (see [38]), set

$$(4.3) \quad \langle f_1 \sqrt{d\mu_1}, f_2 \sqrt{d\mu_2} \rangle = \int_M f_1(x) f_2(x) \sqrt{\frac{d\mu_1}{d\lambda}(x) \frac{d\mu_2}{d\lambda}(x)} d\lambda(x),$$

where  $\lambda$  is chosen such that  $\mu_i \ll \lambda$  for  $i = 1, 2$  (for example, one can take  $\lambda = \mu_1 + \mu_2$ ) and set

$$(4.4) \quad f_1 \sqrt{d\mu_1} + f_2 \sqrt{d\mu_2} = \text{equivalence class of } (f_1 \sqrt{\frac{d\mu_1}{d\lambda}} + f_2 \sqrt{\frac{d\mu_2}{d\lambda}}, \lambda).$$

The operations defined in (4.3) and (4.4) are known to respect the equivalence relation (4.2). The set of all corresponding equivalence classes becomes a Hilbert space, which we shall denote  $\mathcal{H} = \mathcal{H}(M, \mathcal{B})$ . A separate argument is needed in proving completeness, see [38]: If  $(f_n \sqrt{d\mu_n})_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{H}(M, \mathcal{B})$ , there is a pair  $(f, \mu)$  with associated equivalence class  $f \sqrt{d\mu}$  such that

$$\lim_{n \rightarrow \infty} \|f \sqrt{d\mu} - f_n \sqrt{d\mu_n}\|_{\mathcal{H}(M, \mathcal{B})} = 0.$$

**Proposition 4.2.** *The Gaussian processes from Definition 2.1, with  $(M, \mathcal{B}, \mu)$  given, are special cases of the one from Definition 2.2 if we take  $\mathcal{H} = \mathbf{L}^2(\mu)$ .*

**Proof:** To see this, fix  $(M, \mathcal{B}, \mu)$ , and let  $W^{(\mu)}$  be the associated Gaussian process. Then the map

$$(4.5) \quad A \in \mathcal{B}, \mu(A) < \infty \implies W_A^{(\mu)} \in \mathbf{L}^2(\Omega, \mathbb{P})$$

extends to all of  $\mathbf{L}^2(\mu)$ . The extended map, denoted by

$$(4.6) \quad W^{(\mu)}(f) = \int_M f(x) dW_x^{(\mu)},$$

and with range in  $\mathbf{L}^2(\Omega, \mathbb{P})$ , is the Ito integral [22]. When  $f \in \mathbf{L}^2(\mu)$  is a simple function, that is a finite sum of the form

$$(4.7) \quad f = \sum_{i=1}^N a_i \chi_{A_i},$$

where the  $a_i$  are real numbers, and the  $A_i$  belong to  $\mathcal{B}$  are such that  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then set

$$(4.8) \quad W^{(\mu)}(f) = \sum_{i=1}^N a_i W_{A_i}^{(\mu)}.$$

Using

$$(4.9) \quad \mathbb{E}(W_A^{(\mu)} W_B^{(\mu)}) = \mu(A \cap B), \quad \forall A, B \in \mathcal{B},$$

we get

$$(4.10) \quad \mathbb{E}(|W^{(\mu)}(f)|^2) = \sum_{i=1}^N a_i^2 \mu(A_i) = \int_M f(x)^2 d\mu(x).$$

(In the complex case, we use  $\int_M |f(x)|^2 d\mu(x)$  on the right hand-side of (4.10)). Since every function  $f \in \mathbf{L}^2(\mu)$  is the limit (in the norm of

$\mathbf{L}^2(\mu)$ ) of a sequence of simple functions, we conclude that the isometry (4.10) extends to all of  $\mathbf{L}^2(\mu)$ . Furthermore, by polarization,

$$(4.11) \quad \mathbb{E} \left( W^{(\mu)}(f) W^{(\mu)}(g) \right) = \langle f, g \rangle_{\mathbf{L}^2(\mu)}, \quad \forall f, g \in \mathbf{L}^2(\mu).$$

□

**Lemma 4.3.** *Consider  $(M, \mathcal{B})$  as in Definition 4.1, and let  $(f_i, \mu_i)$ ,  $i = 1, 2$  be a pair, see (4.1). Then,*

$$(4.12) \quad (f_1, \mu_1) \sim (f_2, \mu_2) \iff W^{(\mu_1)}(f_1) = W^{(\mu_2)}(f_2), \quad Q \text{ a.e.}$$

**Proof:** We first assume that  $(f_1, \mu_1) \sim (f_2, \mu_2)$ . There exists  $\lambda \in \mathcal{M}(M, \mathcal{B})$  such that both  $\mu_1$  and  $\mu_2$  are absolutely continuous with respect to  $\lambda$  and such that (4.2) is in force. Then,

$$(4.13) \quad \begin{aligned} W^{(\mu_1)}(f_1) &= W^{(\lambda)} \left( f_1 \sqrt{\frac{d\mu_1}{d\lambda}} \right) \\ &= W^{(\lambda)} \left( f_2 \sqrt{\frac{d\mu_2}{d\lambda}} \right) \\ &= W^{(\mu_2)}(f_2), \end{aligned}$$

which is the desired identity on the right hand-side of (4.12). For the justification of (4.13), see Section 3, especially Corollary 3.4.

Conversely, assume that  $W^{(\mu_1)}(f_1) = W^{(\mu_2)}(f_2)$  (almost everywhere with respect to  $Q$ ) for some pairs  $(f_1, \mu_1)$  and  $(f_2, \mu_2)$ . By the argument above applied to  $\lambda = \mu_1 + \mu_2$ , we get

$$(4.14) \quad W^{(\lambda)} \left( f_1 \sqrt{\frac{d\mu_1}{d\lambda}} \right) = W^{(\lambda)} \left( f_2 \sqrt{\frac{d\mu_2}{d\lambda}} \right).$$

Hence, for every  $\varphi \in \mathbf{L}^2(\lambda)$  we have

$$\begin{aligned} \int_M \varphi(x) \left( f_1(x) \sqrt{\frac{d\mu_1}{d\lambda}}(x) - f_2(x) \sqrt{\frac{d\mu_2}{d\lambda}}(x) \right) d\lambda(x) &= \\ &= \mathbb{E}_Q \left( W^{(\lambda)}(\varphi) \underbrace{W^{(\lambda)} \left( f_1 \sqrt{\frac{d\mu_1}{d\lambda}} - f_2 \sqrt{\frac{d\mu_2}{d\lambda}} d\lambda \right)}_{=0} \right) \\ &= 0, \end{aligned}$$

as follows from (4.14). Since this holds for all  $\varphi \in \mathbf{L}^2(\lambda)$  we conclude that

$$f_1 \sqrt{\frac{d\mu_1}{d\lambda}} = f_2 \sqrt{\frac{d\mu_2}{d\lambda}}, \quad \lambda \text{ a.e.},$$

that is  $(f_1, \mu_1) \sim (f_2, \mu_2)$ . □

## 5. THE FIRST MAIN THEOREM

Starting with a measure space  $M$  and a Borel sigma-algebra  $\mathcal{B}$ , we get for every sigma-finite measure  $\mu$  on  $M$  an associated Gaussian process  $W^{(\mu)}$ . Now, for every function  $f \in \mathbf{L}^2(\mu)$ , we may therefore compute an associated Ito-integral of  $f$  with respect to this Gaussian process  $W^{(\mu)}$ ; see Proposition 4.2. We denote this Ito-integral by  $W^{(\mu)}(f)$ . We proved in Section 4 that, when  $f$  and  $\mu$  are given, then the Gaussian random variable  $W^{(\mu)}(f)$  depends only on the equivalence class of the pair  $(f, \mu)$ . As a result we are able to show (Theorem 5.3) that all the Gaussian processes  $W^{(\mu)}$  merge together (via a sigma-Hilbert space) to yield a single Gaussian Hilbert space in the sense of Definition 2.2.

**Definition 5.1.** *Let  $(M, \mathcal{B})$  be fixed, and let  $\mathcal{H}$  denote the corresponding Hilbert space of sigma-functions; see Definition 4.1. For  $\mu \in (M, \mathcal{B})$  we set*

$$(5.1) \quad \mathcal{H}(\mu) = \left\{ f \sqrt{d\mu} \mid f \in \mathbf{L}^2(d\mu) \right\},$$

and

$$(5.2) \quad \mathcal{H}_1(\mu) = \left\{ f \sqrt{d\mu} \mid f \in \mathbf{L}^2(d\mu), |f| \leq 1 \text{ } \mu \text{ a.e.} \right\},$$

**Lemma 5.2.** *Let  $\mu \in \mathcal{M}(M, \mathcal{B})$  be fixed. Then the map*

$$(5.3) \quad Tf = f \sqrt{d\mu}$$

*defines an isometrically isomorphic from  $\mathbf{L}^2(M, \mu)$  onto  $\mathcal{H}(\mu)$*

**Proof:** It follows from Definition 4.1 that  $T$  is isometric. We claim that it is onto. Indeed, a pair  $(g, \nu)$  is in  $\mathcal{H}(\mu)$  if and only if  $\lambda = \mu + \nu$  satisfies

$$(5.4) \quad g \sqrt{\frac{d\nu}{d\lambda}} = f \sqrt{\frac{d\mu}{d\lambda}}, \quad \text{a.e. } \lambda.$$

We claim that

$$(5.5) \quad T^*(g, \nu) = f,$$

where  $f$  is as in (5.4). Indeed, for all  $\varphi \in \mathbf{L}^2(\mu)$  we have

$$\begin{aligned}\langle T\varphi, (g, \nu) \rangle_{\mathcal{H}(\mu)} &= \int_M \varphi(x)g(x) \sqrt{\frac{d\nu}{d\lambda} \frac{d\mu}{d\lambda}}(x) d\lambda(x) \\ &= \int_M \varphi(x)f(x) \frac{d\mu}{d\lambda}(x) d\lambda(x) \\ &= \int_M \varphi(x)f(x) d\mu(x),\end{aligned}$$

and so  $T^*(g, \nu) = f$  as claimed, and  $TT^* = \text{Id}_{\mathcal{H}(\mu)}$ .  $\square$

**Theorem 5.3.** *Let  $\mathcal{H}$  be the sigma-Hilbert space of Definition 4.1. Let  $(M, \mathcal{B})$  be as in Section 2. Then, the map*

$$(5.6) \quad f\sqrt{d\mu} \longrightarrow W^{(\mu)}(f),$$

*defined for every  $\mu \in \mathcal{M}(M, \mathcal{B})$  and  $f \in \mathbf{L}^2(d\mu)$ , extends to an isometry  $F \mapsto \widetilde{W}(F)$  from  $\mathcal{H}$  into  $\mathbf{L}^2(\Omega_s, Q)$ . Furthermore,  $\{\widetilde{W}(F)\}_{F \in \mathcal{H}}$  is a Gaussian  $\mathcal{H}$ -process in the sense of Definition 2.2, i.e.,*

$$(5.7) \quad \mathbb{E}_Q(\widetilde{W}(F_1)\widetilde{W}(F_2)) = \langle F_1, F_2 \rangle_{\mathcal{H}}, \quad \forall F_1, F_2 \in \mathcal{H}.$$

**Proof:** Suppose that  $F_i = f_i\sqrt{d\mu_i}$ ,  $i = 1, 2$ . Then, defining

$$\widetilde{W}(F_i) = W^{(\mu_i)}(f_i), \quad i = 1, 2,$$

(see (5.6)), identity (5.7) holds. Indeed,

$$\begin{aligned}\mathbb{E}_Q(\widetilde{W}(F_1)\widetilde{W}(F_2)) &= \mathbb{E}_Q(W^{(\mu_1)}(f_1)W^{(\mu_2)}(f_2)) \\ &= \mathbb{E}_Q\left(W^{(\lambda)}\left(f_1\sqrt{\frac{d\mu_1}{d\lambda}}\right)W^{(\lambda)}\left(f_2\sqrt{\frac{d\mu_2}{d\lambda}}\right)\right) \\ &= \int_M f_1(x)f_2(x)\sqrt{\frac{d\mu_1}{d\lambda}(x)\frac{d\mu_2}{d\lambda}(x)}d\lambda(x) \\ &= \langle F_1, F_2 \rangle_{\mathcal{H}},\end{aligned}$$

where, in the last step, we used (4.3) in the definition of the inner product in  $\mathcal{H}$ .



We now turn to the linearity of  $\widetilde{W}$ . For the sum in  $\mathcal{H}$  we have equation (4.4). Hence,

$$\begin{aligned}\widetilde{W}(F_1 + F_2) &= W^{(\lambda)} \left( f_1 \sqrt{\frac{d\mu_1}{d\lambda}} + f_2 \sqrt{\frac{d\mu_2}{d\lambda}} \right) \\ &= W^{(\lambda)} \left( f_1 \sqrt{\frac{d\mu_1}{d\lambda}} \right) + W^{(\lambda)} \left( f_2 \sqrt{\frac{d\mu_2}{d\lambda}} \right) \\ &= \widetilde{W}(F_1) + \widetilde{W}(F_2).\end{aligned}$$

It remains to prove that  $\widetilde{W}(\cdot)$  satisfies the joint Gaussian property stated in Remark 2.3. We must prove that if  $F_i = f_i \sqrt{d\mu_i}$ ,  $i = 1, 2, \dots, n$ , then the joint distribution of

$$(\widetilde{W}(F_1), \widetilde{W}(F_2), \dots, \widetilde{W}(F_n))$$

is the Gaussian random variable in  $\mathbb{R}^n$  with zero mean and covariance matrix  $(\langle F_i, F_j \rangle_{\mathcal{H}})_{i,j=1}^n$ . To see this, pick  $\lambda \in \mathcal{M}(M, \mathcal{B})$  such that  $\mu_1, \mu_2, \dots, \mu_n$  are all absolutely continuous with respect to  $\lambda$  (for instance,  $\lambda = \sum_{i=1}^n \mu_i$ ). Then, in view of (4.3),

$$(5.8) \quad \langle F_i, F_j \rangle_{\mathcal{H}} = \int_M f_i(x) f_j(x) \sqrt{\frac{d\mu_i}{d\lambda}(x) \frac{d\mu_j}{d\lambda}(x)} d\lambda(x).$$

But,

$$\begin{aligned}\mathbb{E}_Q \left( \widetilde{W}(F_i) \widetilde{W}(F_j) \right) &= \mathbb{E}_Q \left( W^{(\mu_i)}(f_i) W^{(\mu_j)}(f_j) \right) \\ &= \mathbb{E}_Q \left( W^{(\lambda)} \left( f_i \sqrt{\frac{d\mu_i}{d\lambda}} \right) W^{(\lambda)} \left( f_j \sqrt{\frac{d\mu_j}{d\lambda}} \right) \right) \\ &= \langle f_i \sqrt{\frac{d\mu_i}{d\lambda}}, f_j \sqrt{\frac{d\mu_j}{d\lambda}} \rangle_{L^2(\lambda)},\end{aligned}$$

which is equal to the right hand-side of (5.8), and leads to the desired conclusion.  $\square$

We conclude this section with:

**Proposition 5.4.**  *$f \in \mathcal{H}_1(\mu)$  if and only if  $f$  is the correlation function for two copies of  $W^{(\mu)}$ .*

**Proof:** One direction is clear. Let  $W_1^{(\mu)}$  and  $W_2^{(\mu)}$  be two correlated copies of  $W^{(\mu)}$  and set

$$\nu(A) = \mathbb{E} \left( (W_1)_A^{(\mu)} (W_2)_A^{(\mu)} \right).$$

$\nu$  is a signed measure, defining the correlation between the two copies of  $W^{(\mu)}$ . By a Cauchy-Schwarz inequality,  $\nu$  is absolutely continuous with respect to  $\mu$ . One then checks that the Radon-Nikodym  $f$  belongs to  $\mathcal{H}_1(\mu)$ . Conversely, given  $f \in \mathcal{H}_1(\mu)$ , it suffices to define a signed measure by

$$\nu(A) = \int_A f(x) d\mu(x)$$

to define two correlated copies of  $W^{(\mu)}$ . □

## 6. REPRESENTATION OF $W^{(\mu)}$ IN AN ARBITRARY PROBABILITY SPACE $(\Omega, \mathcal{F}, P)$

In this section we prove that the infinite-product measure space (Theorem 3.3) is universal in the sense that every measure space  $(\Omega, \mathcal{F}, P)$  which carries some Gaussian processes  $W^{(\mu)}$ , i.e., makes  $W^{(\mu)}$  into an  $\mathbf{L}^2$  Gaussian process, can be computed directly from the universal infinite-product measure space. This is spelled out in Theorems 6.1 in this section and in Theorem 7.1 in the next section.

In the previous section we have established a decomposition of the Gaussian process  $W^{(\mu)}$  as an expansion in a system of i.i.d.  $N(0, 1)$  random variables. As before  $(M, \mathcal{B}, \mu)$  is a given sigma-finite measure space.

**Theorem 6.1.** *Let  $W^{(\mu)}$  be represented in a probability space  $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ , see Definition 2.1, and let  $\{\varphi_j\}_{j \in \mathbb{N}}$  be an orthonormal basis in  $\mathbf{L}^2(\mu)$ . Then, there is a system  $\{Z_j\}_{j \in \mathbb{N}}$  of i.i.d.  $N(0, 1)$  random variables such that*

$$(6.1) \quad W_A^{(\mu)} = \sum_{j=1}^{\infty} \left( \int_A \varphi_j(t) d\mu(t) \right) Z_j(\cdot)$$

*holds almost everywhere on  $\Omega$  with respect to  $\mathbb{P}$ .*

**Proof:** Assume that  $A \in \mathcal{B}$  and  $0 < \mu(A) < \infty$ . We proceed in a number of steps.

STEP 1: *The system  $\{Z_j\}_{j \in \mathbb{N}}$*

$$(6.2) \quad Z_j = W^{(\mu)}(\varphi_j) = \int_M \varphi_j(x) dW_x^{(\mu)} \quad (\text{as an Ito integral})$$

is a family of i.i.d.  $N(0, 1)$  variables.

To see this, we use the construction in Proposition 4.2. Indeed,

$$\begin{aligned}\mathbb{E}(Z_j Z_k) &= \mathbb{E}(W^{(\mu)}(\varphi_j) W^{(\mu)}(\varphi_k)) \\ &= \langle \varphi_j, \varphi_k \rangle_{\mathbf{L}^2(\mu)} \\ &= \delta_{j,k}, \quad \forall j, k \in \mathbb{N},\end{aligned}$$

are the desired orthogonality condition. The rest of the assertion is clear.

STEP 2: We show that the sum on the right-hand-side of (6.2) converges in the norm of  $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$  and

$$\begin{aligned}\mathbb{E} \left( \left| W_A^{(\mu)} - \sum_{j=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right) Z_j \right|^2 \right) &= \mathbb{E} \left( |W_A^{(\mu)}|^2 \right) - \\ &\quad - 2 \sum_{j=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right) \mathbb{E} \left( W_A^{(\mu)} Z_j \right) + \\ &\quad + \sum_{j=1}^{\infty} \left| \int_A \varphi_j(x) d\mu(x) \right|^2 \\ &= \mu(A) - \sum_{j=1}^{\infty} \left| \int_A \varphi_j(x) d\mu(x) \right|^2 \\ &= \mu(A) - \|\chi_A\|_{\mathbf{L}^2(\mu)}^2 \\ &= \mu(A) - \mu(A) = 0.\end{aligned}$$

□

**Corollary 6.2.** *Consider the space  $(M, \mathcal{B}, \mu)$  as in the previous theorem, and let  $W^{(\mu)}$  be represented in some probability space  $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ . Then, some point  $x_0 \in M$  is an atom, i.e.  $\mu(\{x_0\}) > 0$ , where  $\{x_0\}$  denotes the singleton, if and only if the ONB  $\{\varphi_j\}_{j \in \mathbb{N}}$  in  $\mathbf{L}^2(\mu)$  has  $\varphi(x_0)$  well defined, the expansion (6.2) contains a term*

$$(6.3) \quad \mu(\{x_0\}) \sum_{j=1}^{\infty} \varphi(\{x_0\}) Z_j,$$

and

$$(6.4) \quad \sum_{j=1}^{\infty} (\varphi_j(x_0))^2 = \frac{1}{\mu(\{x_0\})}$$

**Proof:** Functions  $f \in \mathbf{L}^2(\mu)$  are determined only point-wise a.e with respect to  $\mu$ , but if  $\mu(\{x_0\}) > 0$ , the functions  $f$  are necessarily well defined at the point  $x_0$ , i.e.,  $f(x_0)$  is a uniquely defined finite number. We apply this to the functions  $\varphi_j$  in the  $\mathbf{L}^2(\mu)$ -ONB from (6.2). Hence, the contributions to the two sides in (6.2) corresponding to  $A = \{x_0\} \in \mathcal{B}$  are as follows:

$$(6.5) \quad W_{\{x_0\}}^{(\mu)}(\cdot) = \mu(\{x_0\}) \sum_{j=1}^{\infty} \varphi(\{x_0\}) Z_j(\cdot).$$

Taking norms in  $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$  we get

$$\mu(\{x_0\}) = (\mu(\{x_0\}))^2 \sum_{j=1}^{\infty} (\varphi(x_0))^2,$$

and the desired conclusion (6.4) follows.  $\square$

**Remark 6.3.** *Some care must be exercised in assigning the random variable  $W_A^{(\mu)}$  to sets  $A \in \mathcal{B}$  with  $\mu(A) = 0$ , or  $\mu(A) = \infty$ : If  $\mu(A) = 0$ , we may take  $W_A^{(\mu)}$  to have law the Dirac distribution  $\delta_0$  on  $\mathbb{R}$  at  $x = 0$ . In view of (6.1) one may alternatively set  $W_A^{(\mu)} = 0$  if  $\mu(A) = 0$ . There are two conventions for dealing with the random variable  $X_A^{(\mu)}$  when  $\mu(A) = \infty$ . One involves a renormalization, somewhat subtle. For other purposes, if  $\mu(A) = \infty$ , we may simply take the random variable  $W_A^{(\mu)}$  to have the uniform distribution.*

We now explain the connections between the present construction and the processes we built in [6, 5].

**Application 6.4.** *Let  $\mathcal{S}$  denote the Schwartz space of smooth functions on  $\mathbb{R}$  with its Fréchet topology and let  $\mu$  be a Borel measure on  $\mathbb{R}$  such that  $\int_{\mathbb{R}} \frac{d\mu(t)}{(1+t^2)^p} < \infty$  for some  $p \in \mathbb{N}_0$  (where  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ). Then:*  
(i) *The function  $F$ :*

$$F(\varphi) = e^{-\frac{1}{2} \int_{\mathbb{R}} |\widehat{\varphi}(t)|^2 d\mu(t)}$$

(where  $\widehat{\varphi}$  denotes the Fourier transform of  $\varphi$ ) is positive definite and continuous from  $\mathcal{S}$  into  $\mathbb{R}_+$ , in the Fréchet topology. By Minlos' theorem there exists a uniquely defined probability measure  $\mathbb{P}$  on the space  $\mathcal{S}'$  of tempered distributions such that

$$F(\varphi) = \mathbb{E}_{\mathbb{P}} \left( e^{i\langle \cdot, \varphi \rangle} \right).$$

(ii) Furthermore we showed that there is a Gaussian process on  $\mathcal{S}'$  with the Wiener measure such that

$$F(\varphi) = \mathbb{E}_{\text{Wiener}} \left( e^{iX_{\varphi}^{(\mu)}} \right).$$

From the results of the present paper, we then get

$$F(\varphi) = \mathbb{E}_Q \left( e^{iW^{(\mu)}(\widehat{\varphi})} \right),$$

where  $Q$  is the probability measure defined in Lemma 3.1, and where the process  $W^{(\mu)}$  is constructed in Proposition 4.2 and Theorem 3.3.

In summary, the two Gaussian processes  $X^{(\mu)}(\varphi)$  and  $W^{(\mu)}(\widehat{\varphi})$  have the same generating function.

As a corollary we have:

**Corollary 6.5.** *Let  $\lambda = dx$  denote the Lebesgue measure  $dx$  on the real line and the Gaussian processes  $W^{(\mu)}(\varphi)$  constructed from measures  $\mu$  such that  $\mu \ll \lambda$  include the fractional Brownian motion. We get this from the choice  $d\mu(x) = c_H |x|^{2H} dx$  where  $H \in (0, 1)$  and  $c_H$  is some appropriate constant.*

For some recent work on the fractional Brownian motion, see also [1, 8, 32, 36].

## 7. THE PROBABILITY SPACE $(\times_{\mathbb{N}} \mathbb{R}, \mathcal{F}, Q_K)$ IS UNIVERSAL

Suppose a Gaussian processes  $W^{(\mu)}$  is represented in some measure space  $(\Omega, \mathcal{F}, P)$ , we will then be able to compute the measure  $P$ , and study how it depends on the initial measure  $\mu$  on  $M$ . This we do in Theorem 7.1 below, which also yields a measure-isomorphism connecting  $P$  to an infinite-product measure.

In this section we will show that when  $(M, \mathcal{B}, \mu)$  is given as above, that is, is some fixed sigma-finite measure space, then every realization of the corresponding Gaussian process  $W^{(\mu)}$  factors through  $(\times_{\mathbb{N}} \mathbb{R}, Q_K)$ . More precisely suppose that  $W^{(\mu)}$  is realized as a Gaussian  $\mathbf{L}^2$ -process in some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , then there is a factor-mapping

setting up to an isomorphism of the respective Gaussian processes on  $(\Omega, \mathcal{F}, \mathbb{P})$  and on  $(\times_{\mathbb{N}}\mathbb{R}, Q_K)$ .

**Theorem 7.1.** *Let  $(M, \mathcal{B}, \mu)$  be fixed, and let the associated process (see Definition 2.1 and Theorem 3.3) be realized in  $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ , where  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space. Suppose*

$$(7.1) \quad \mathcal{F} = \sigma - \text{alg.} \left\{ W_A^{(\mu)} \mid A \in \mathcal{B} \right\}.$$

*Then, the following assertions hold:*

(i) *For all  $A \in \mathcal{B}$  with  $0 < \mu(A) < \infty$  and  $a, b \in \mathbb{R}$  (with  $a < b$ ) we have*

$$(7.2) \quad \mathbb{P} \left( \left\{ \omega \in \Omega \mid a < W_A^{(\mu)}(\omega) \leq b \right\} \right) = \gamma_1 \left( \left( \frac{a}{\sqrt{\mu(A)}}, \frac{b}{\sqrt{\mu(A)}} \right] \right),$$

*where  $\gamma_1$  is the standard  $N(0, 1)$ -Gaussian.*

(ii) *There is a measure isomorphism*

$$\Psi : \Omega \longrightarrow \times_{\mathbb{N}}\mathbb{R}$$

*such that*

$$(7.3) \quad \mathbb{P} \circ \Psi^{-1} = Q_K \quad \text{and} \quad W^{(Q_K, \mu)} \circ \Psi = W^{(\mu)}$$

*hold almost everywhere on  $\Omega$ , and where  $W^{(Q_K, \mu)}$  denotes the realization of  $W^{(\mu)}$  on  $(\times_{\mathbb{N}}\mathbb{R}, Q_K)$  from Section 4.*

**Proof:** Since  $W_A^{(\mu)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ , it follows that every cylinder set specified as in (i), i.e.,

$$\left\{ a < W_A^{(\mu)} \leq b \right\},$$

belongs to  $\mathcal{F}$ . since  $W_A^{(\mu)}$  is a Gaussian variable with law  $N(0, \mu(A))$ , formula (7.2) from (i) must hold. Now pick an ONB  $\{\varphi_j\}_{j \in \mathbb{N}}$  in  $\mathbf{L}^2(\mu)$  and, following Theorem 3.3, set

$$Z_j := W^{(\mu)}(\varphi_j), \quad j \in \mathbb{N}.$$

Then,  $\{Z_j\}_{j \in \mathbb{N}}$  is an i.i.d.  $N(0, 1)$  family, and (6.2) holds. Now define

$$\Psi : \Omega \longrightarrow \times_{\mathbb{N}}\mathbb{R}$$

by

$$(7.4) \quad \Psi(\omega) = (Z_j(\omega))_{j \in \mathbb{N}},$$

or equivalently,

$$(7.5) \quad \pi_j \circ \Psi = Z_j, \quad \forall j \in \mathbb{N}.$$

Applying (7.5) to the expansion (6.2) for  $W^{(\mu)}$  and for  $W^{(Q_K, \mu)}$ , we see get

$$W^{(Q_K, \mu)} \circ \Psi = W^{(\mu)},$$

that is, the stochastic process  $W^{(\mu)}$  factors as stated.

Using again (6.2) from Theorem 6.1, we see that  
(7.6)

$$\Psi \left( \left\{ a < W_A^{(\mu)} \leq b \right\} \right) \subseteq \left\{ (\xi_j)_{j \in \mathbb{N}} \in \times_{\mathbb{N}} \mathbb{R} \mid a < \sum_{j=1}^{\infty} \xi_j \int_A \varphi_j(x) d\mu(x) \leq b \right\},$$

and that

$$\begin{aligned} Q \left( \Psi \left( \left\{ a < W_A^{(\mu)} \leq b \right\} \right) \right) &= \gamma_1 \left( \left( \frac{a}{\sqrt{\mu(A)}}, \frac{b}{\sqrt{\mu(A)}} \right] \right) \\ &= \gamma_A((a, b]) \\ &= \mathbb{P} \left( \left\{ a < W_A^{(\mu)} \leq b \right\} \right), \end{aligned}$$

where we used (7.2) in the last step of the reasoning. Since  $\mathcal{F}_s$  is generated (as a sigma-algebra) by the cylinder sets, the final assertion  $\mathbb{P} \circ \Psi^{-1} = Q_K$  in (ii) follows. We do this by passing from monic subsets  $\left\{ a < W_A^{(\mu)} \leq b \right\}$ , to finite functions, and to measurable functions by inductive limit.

A function  $F$  on  $(\Omega, \mathcal{F})$  is said to be finite if there is  $n \in \mathbb{Z}_+$ , a bounded  $\mathbb{R}^n$ -Borel function  $f_n$ , and  $A_1, \dots, A_n \in \mathcal{B}$  such that

$$(7.7) \quad F(\cdot) = f_n(W_{A_1}^{(\mu)}(\cdot), \dots, W_{A_n}^{(\mu)}(\cdot)).$$

With  $F$  as in (7.7), we then have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(F) &= \int_{\Omega} f_n(W_{A_1}^{(\mu)}(\omega), \dots, W_{A_n}^{(\mu)}(\omega)) d\mathbb{P}(\omega) \\
&= \underbrace{\int \cdots \int}_{\mathbb{R}^n} f_n(x_1, x_2, \dots, x_n) \times \\
&\quad \times \gamma_{A_1}(x_1) \gamma_{A_2}(x_2 - x_1) \cdots \gamma_{A_n}(x_n - x_{n-1}) dx_1 dx_2 \cdots dx_n \\
&= \underbrace{\int \cdots \int}_{\mathbb{R}^n} f_n(y_1, y_1 + y_2, \dots, y_1 + y_2 + \cdots + y_n) \times \\
&\quad \times \gamma_{A_1}(y_1) \gamma_{A_2}(y_2) \cdots \gamma_{A_n}(y_n) dy_1 dy_2 \cdots dy_n \\
&= \int_{\mathbf{s}'} f_n(W_{A_1}^{(Q_K, \mu)}, \dots, W_{A_n}^{(Q_K, \mu)}) dQ_K \\
&= \int_{\Omega} f_n(W_{A_1}^{(\mu)}, \dots, W_{A_n}^{(\mu)}) d(Q \circ \Psi) \\
&= \mathbb{E}_{Q_K \circ \Psi}(F).
\end{aligned}$$

□

**Corollary 7.2.** Fix  $(M, \mathcal{B}, \mu)$  as in Theorems 6.1 and 7.1, and denote by  $\Omega_{\mathcal{B}}$  the set of all finitely additive functions  $\omega : \mathcal{B} \longrightarrow \mathbb{R}$ . Set

$$(7.8) \quad W_A(\omega) = \omega(A), \quad \forall A \in \mathcal{B},$$

and let  $\{Z_j\}_{j \in \mathbb{N}}$  be the corresponding i.i.d.  $N(0, 1)$  system from Theorem 6.1 (The measure  $\mathbb{P}$  on  $\omega_{\mathcal{B}}$  is from (7.3) in Theorem 7.1, i.e.

$$\mathbb{P} \circ \Psi^{-1} = Q_K,$$

where  $\Psi$  is given by (7.4), that is,  $\Psi(\omega) = (Z_j(\omega))_{j \in \mathbb{N}}$ ,  $\forall \omega \in \mathcal{B}$ ). For  $\xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbf{s}'$ , and  $A \in \mathcal{B}$ , set

$$(7.9) \quad \Gamma(\xi)(A) := \sum_{j=1}^{\infty} \xi_j \int_A \varphi_j(x) d\mu(x).$$

Then,

- (i)  $\Gamma(\xi) \in \Omega_{\mathcal{B}}$
- (ii)  $\Psi(\Gamma(\xi)) = \xi, \quad \forall \xi \in \mathbf{s}',$

and

- (iii)  $\mathbb{P}(\{\omega \in \Omega_{\mathcal{B}}, \mid \Gamma(\Psi(\omega)) = \omega\}) = 1$  .



**Proof:** The asserted conclusions follow from Theorem 6.1 and 7.1. Note that (iii) in the Corollary says that

$$(7.10) \quad \Gamma \circ \Psi = \text{Id}_{\Omega_{\mathcal{B}}}, \quad \mathbb{P}_{\mathcal{B}} \text{ a.e.}$$

where  $\mathbb{P}_{\mathcal{B}}$  is the measure on  $\Omega_{\mathcal{B}}$  given by

$$(7.11) \quad \mathbb{P}_{\mathcal{B}} \circ \Psi^{-1} = Q_K.$$

Now, formula (6.1) is an identity in  $\mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P}_{\mathcal{B}})$ . Since

$$W_A^{(\mathcal{B})}(\omega) = \omega(A), \quad \forall \omega \in \Omega_{\mathcal{B}},$$

we get the following  $\mathbb{P}_{\mathcal{B}}$ -a.e. identity holding on  $\Omega_{\mathcal{B}}$ :

$$\begin{aligned} \omega(A) &= W_A^{(\mathcal{B})}(\omega) \\ &= \sum_{j=1}^{\infty} \left( \int_A \varphi_j(x) d\mu(x) \right) Z_j(\omega) \\ &= ((\Gamma \circ \Psi)(\omega))(A), \quad \forall A \in \mathcal{B}. \end{aligned}$$

This proves (7.10). □

**Corollary 7.3.** *Let  $\Psi : \Omega \longrightarrow \prod_{\mathbb{N}} \mathbb{R}$  be as in (7.4), and define the induced operator  $A$  from the bounded Borel function defined on  $\prod_{\mathbb{N}} \mathbb{R}$  into the bounded Borel function defined on  $\Omega$ ,*

$$Af = f \circ \Psi.$$

*Then,  $A$  is a Markov operator (see [10]), i.e. the following properties hold:*

$$(i) \quad f \geq 0 \text{ a.e.} \implies Af \geq 0, \text{ a.e.},$$

$$(ii) \quad A\mathbf{1} = \mathbf{1},$$

$$(iii) \quad A^*\mathbf{1} = \mathbf{1}.$$

**Proof:** Note that in (ii) and (iii) the symbol  $\mathbf{1}$  denote the constant function equal to 1 in the respective measured spaces. Properties (i) and (ii) are clear. For (iii) we use the fact that functions of the form

$$F(\xi) = f_n(\xi_1, \dots, \xi_n).$$

where  $f_n$  is a bounded Borel function on  $\mathbb{R}^n$  are  $\mathbf{L}^2$  dense. For such a function, we want to check that

$$\mathbb{E}_{Q_K}(F(\mathbf{1} - A^*\mathbf{1})) = 0,$$

or, equivalently,

$$\mathbb{E}_P(A(F)) = \mathbb{E}_{Q_T}(F).$$

We have

$$\begin{aligned} \mathbb{E}_P(A(F)) &= \mathbb{E}(f_n(Z_1(\cdot), \dots, Z_n(\cdot))) \\ &= \underbrace{\int \int \cdots \int}_{\mathbb{R}^n} f_n(x_1, \dots, x_n) \gamma_n(x_1, \dots, x_n) dx_1 \cdots dx_n \\ &= \mathbb{E}_{Q_K}(F), \end{aligned}$$

where we have used the properties from Theorem 7.1 for the respective measures  $P$  and  $Q_K$ , as well as the i.i.d. system  $(Z_j)_{j \in \mathbb{N}}$  from (6.2) in Theorem 6.1.  $\square$

**Corollary 7.4.** *Let  $(M, \mathcal{B}, \mu)$ ,  $\Omega_{\mathcal{B}}$  and  $\mathbb{P} := \mathbb{P}_{\mathcal{B}}$  be as in Corollary 7.2. Set on  $\mathcal{B} \times \mathcal{B}$*

$$(7.12) \quad K(A, B) = e^{\left(\mu(A \cap B) - \frac{\mu(A) + \mu(B)}{2}\right)}.$$

*Then we may define a Fourier transform  $F \mapsto \hat{F}$  from  $\mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$  onto the reproducing kernel Hilbert space  $\mathcal{H}(K)$  with reproducing kernel  $K$  as in (7.12). For  $F \in \mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$ ,*

$$(7.13) \quad \hat{F}(A) = \mathbb{E}\left(F(\cdot) e^{iW_A^{(\mu)}(\cdot)}\right), \quad A \in \mathcal{B}.$$

*Moreover the map  $F \mapsto \hat{F}$  is an isometric isomorphism between the two Hilbert spaces.*

**Proof:** We begin with finite sums of the form  $\sum_{j \in J} a_j K_{A_j}$ , where the  $a_j$  are real,  $A_j \in \mathcal{B}$  and  $|J| < \infty$ . Comparing the Hilbert norms we have

$$\begin{aligned} \left\| \sum_{j \in J} a_j K_{A_j} \right\|_{\mathcal{H}(K)}^2 &= \sum_{j, k \in J} a_j a_k K(A_j, A_k) \\ &= \sum_{j, k \in J} a_j a_k e^{-\frac{1}{2} \|\chi_{A_j} - \chi_{A_k}\|_{\mathbf{L}^2(\mu)}^2} \\ &= \sum_{j, k \in J} a_j a_k \mathbb{E} \left( e^{iW_{A_j}^{(\mu)}} e^{-iW_{A_k}^{(\mu)}} \right) \\ &= \left\| \sum_{j \in J} a_j e^{iW_{A_j}^{(\mu)}} \right\|_{\mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})}^2, \end{aligned}$$

where we have used Theorem 3.3 (see in particular (3.14)) in the last step in the computation. This complete the proof of the isometry since

such finite sums are dense in  $\mathcal{H}(K)$ . Completing by taking  $\mathcal{H}(K)$ -norm closure, we see that the adjoint of the map  $J(F) = \widehat{F}$  is isometric from  $\mathcal{H}(K)$  into  $\mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$ . Indeed,

$$(7.14) \quad J\left(e^{iW_A^{(\mu)}}\right) = K_A \in \mathcal{H}(K),$$

and

$$(7.15) \quad J^*(K_A) = e^{iW_A^{(\mu)}}, \quad A \in \mathcal{B}.$$

It remains to prove that

$$\left\{e^{iW_A^{(\mu)}} \mid A \in \mathcal{B}\right\}$$

span a dense subspace in  $\mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$ , and if  $F \in \mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$  is such that

$$(7.16) \quad \widehat{F}(A) = \mathbb{E}\left(F e^{iW_A^{(\mu)}}\right) = 0, \quad \forall A \in \mathcal{B},$$

then  $F = 0$ .

To verify this, we may use the known representation of  $\mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$  as the symmetric Fock space over  $\mathbf{L}^2(d\mu)$ ; see [20]. We also make use of Theorem 7.1 above. Suppose  $F \in \mathbf{L}^2(\Omega_{\mathcal{B}}, \mathbb{P})$  satisfies (7.16). In the Fock-space representation,

$$(7.17) \quad F = \sum_{n=0}^{\infty} F_n$$

is referring to Wiener chaos expansion of  $F$ , that is, the orthogonal decomposition of  $F$  along the orthogonal sum of all symmetric  $n$ -tensors, as  $n = 0, 1, 2, \dots$ , and with  $n = 0$  referring to the vacuum vector. See also [11]. Substitution of (7.17) into (7.16) yields

$$(7.18) \quad \mathbb{E}\left(F_n \underbrace{W_A^{(\mu)} \times \cdots \times W_A^{(\mu)}}_{n \text{ times}}\right) = 0, \quad \forall A \in \mathcal{B}, \quad \text{and} \quad n = 0, 1, \dots$$

Using now the Ito-integral from Proposition 4.2, equation (7.18) may be rewritten as

$$\underbrace{\int_A \int_A \cdots \int_A}_{n \text{ times}} F_n(x_1, x_2, \dots, x_n) dW_{x_1}^{(\mu)} dW_{x_2}^{(\mu)} \cdots dW_{x_n}^{(\mu)} = 0,$$

that is (and where  $\otimes$  denotes the symmetric tensor product),

$$(7.19) \quad F_n \perp \otimes_1^n \chi_A, \quad \forall A \in \mathcal{B}.$$

Since  $F_n \in \otimes_1^n \mathbf{L}^2(\mu)$  is a symmetric tensor, we conclude from (7.19) that  $F_n = 0$ . This holds for  $n = 0, 1, \dots$  and so by (7.17),  $F = 0$ .  $\square$

**Remark 7.5.** *The fact that  $K(A, B)$  is positive definite on  $\mathcal{B}$  can be checked also as follows: The function*

$$n(A, B) = -\mu(A \cap B) + \frac{\mu(A) + \mu(B)}{2}$$

*is conditionally negative on  $\mathcal{B}$ , and therefore the function  $e^{-n(A, B)}$  is positive definite there. See [12] for the latter.*

## 8. ITERATED FUNCTION SYSTEMS

The purpose of the present section is to give an application of the theorems from Sections 6 and 7 to iterated function systems (IFS), see e.g. [26]. Such IFSs arise in geometric measure theory, in harmonic analysis, and in the study of dynamics of iterated substitutions with rational functions (on Riemann surfaces); hence the name iterated function system. With an IFS, we have the initial measure space  $M$  and a Borel sigma-algebra  $\mathcal{B}$ , coming with an additional structure, a system of measurable endomorphisms. We will be interested in those measures  $\mu$  on  $M$  which satisfy suitable self-similarity properties with respect to the prescribed endomorphisms in  $M$ . For background, see e.g. [27, 28, 29].

Given a measure space  $(M, \mathcal{B})$  as in Section 2, i.e.  $\mathcal{B}$  is a fixed Borel sigma-algebra of subsets of  $M$ , by an *iterated function system* (IFS), we mean a system of endomorphisms  $(\tau_i)_{i \in I}$

$$\tau_i : M \longrightarrow M,$$

each  $\tau_i$  assumed measurable and the index set  $I$  usually finite.

If a family of measures  $\mu$  on  $\mathcal{B}$  is specified, each  $\tau_i$  is defined a.e.. Typically,  $M$  will be a locally compact Hausdorff space, and we assume that each  $\tau_i$  is continuous. The following restrictions will be placed on the family  $(\tau_i)_{i \in I}$ :

$$(8.1) \quad \tau_i(M) \cap \tau_j(M) = \emptyset, \quad \forall i, j \in I \quad (\text{non-overlapping}),$$

$$(8.2) \quad \bigcup_{i \in I} \tau_i(M) = M, \quad (\text{cover}),$$

and there is a measurable endomorphism  $R$  from  $M$  into  $M$  such that

$$(8.3) \quad R \circ \tau_i = \text{Id}_M, \quad \forall i \in I.$$

We say that the family  $\{\tau_i\}_{i \in I}$  is a system of branches of an inverse to  $R$ . This is in particular the case in applications to Riemann surfaces, where  $R$  is typically a rational function.

In view of the following definition, recall that we have defined  $(\mu)$  in Definition 5.1.

**Definition 8.1.** Let  $(M, \mathcal{B})$  be fixed, and let  $\mathcal{H}$  denote the corresponding Hilbert space of sigma-functions; see Definition 4.1, and let  $\mu \in (M, \mathcal{B})$ . If  $R : M \rightarrow M$  is a measurable endomorphism, we consider the measure  $\mu \circ R^{-1}$ , i.e.

$$(8.4) \quad (\mu \circ R^{-1})(A) = \mu(R^{-1}(A)), \quad \forall A \in \mathcal{B},$$

where

$$R^{-1}(A) = \{x \in M \mid R(x) \in A\}.$$

We set

$$(8.5) \quad (\mathcal{H} \circ R)(\mu) = \left\{ (f \circ R) \sqrt{d\mu} \mid f \in \mathbf{L}^2(\mu \circ R^{-1}) \right\}.$$

**Definition 8.2.** Let  $\mu \in (M, \mathcal{B})$ , and let  $\{\tau_i\}_{i \in I}$  be as in (8.1)-(8.3) in the previous definition. We say that  $(\mu, \{\tau_i\}_{i \in I})$  is an iterative function system (IFS) if

$$(8.6) \quad \mu \circ \tau_i^{-1} < \mu, \quad \forall i \in I.$$

An IFS is said to be closed if

$$(8.7) \quad \sum_{i \in I} \frac{d(\mu \circ \tau_i^{-1})}{d\mu} = 1.$$

Note that the Radon-Nikodym derivatives in the summation (7.2) are well defined on account of (8.6).

**Remark 8.3.** Special cases of IFS have been widely studied in the literature; see e.g. [27, 28, 29, 33, 42, 41, 47].

In these examples, the Radon-Nikodym derivatives  $\frac{d(\mu \circ \tau_i^{-1})}{d\mu}$  in (8.7) are constant functions, say

$$\frac{d(\mu \circ \tau_i^{-1})}{d\mu} = p_i, \quad i \in I,$$

and  $\sum_{i \in I} p_i = 1$ , so that in particular  $p_i \in (0, 1)$ . As further special cases of this, we have the Cantor measures: For example, let  $M$  be the usual middle third Cantor set, and define two endomorphisms

$$\tau_0(x) = \frac{x}{3}, \quad \text{and} \quad \tau_1(x) = \frac{x+2}{3}.$$

Then, there is a unique probability measure  $\mu$  supported on  $M$  such that

$$(8.8) \quad \mu = \frac{1}{2} (\mu \circ \tau_0^{-1} + \mu \circ \tau_1^{-1}).$$

This is an IFS, and  $p_0 = p_1 = \frac{1}{2}$ ; compare with (8.7). The scaling dimension of  $\mu$  is  $\log_3(2) = \frac{\ln 2}{\ln 3}$ .

**Lemma 8.4.** *Let  $(\mu, \{\tau_i\}_{i \in I})$  is an iterative function system. Then for each  $i \in I$  the mapping*

$$(8.9) \quad (f, d\mu) \mapsto (f \circ R, \mu \circ \tau_i^{-1})$$

*induces (by passing to equivalence classes) an isometry from  $\mathcal{H}(\mu)$  into  $(\mathcal{H} \circ R)(\tau \circ \tau_i^{-1})$ .*

**Proof:** In principle there are issues with passing the transformation onto equivalence classes, but this can be done via an application of Lemma 4.3. Hence in studying (8.9), the question reduces to checking instead that the application

$$(8.10) \quad W^{(\mu)}(f) \mapsto W^{(\mu \circ \tau_i^{-1})}(f \circ R)$$

is isometric. Indeed,

$$\begin{aligned} \mathbb{E}(|W^{(\mu)}(f)|^2) &= \int_M |f(x)|^2 d\mu(x) \\ &= \int_M (|f \circ R \circ \tau_i(x)|^2) d\mu(x) \\ &= \int_M (|(f \circ R)(x)|^2) d(\mu \circ \tau_i^{-1})(x) \\ &= \mathbb{E}_Q(|W^{(\mu \circ \tau_i^{-1})}(f \circ R)|^2), \end{aligned}$$

which is the desired conclusion.  $\square$

We now turn to representation of the Cuntz relations; see e.g. [13, 14, 28].

**Theorem 8.5.** *Let  $(\mu, \{\tau_i\}_{i \in I})$  is a closed iterated function system, and set  $g_i = \frac{d(\mu \circ \tau_i^{-1})}{d\mu}$  (see (8.6) and (8.7)). Then the operators*

$$(8.11) \quad S_i(f) = \chi_{\tau_i(M)} \sqrt{g_i}(f \circ R), \quad i \in I,$$

*define a representation of the Cuntz algebra  $\mathcal{O}_I$  (with index set  $I$ ), acting on the Hilbert space  $\mathbf{L}^2(\mu)$ , i.e. as isometries in  $\mathbf{L}^2(\mu)$ , the*

operators  $S_i$  from (8.11) satisfy:

$$(8.12) \quad S_i^* S_j = \delta_{i,j} \text{Id}_{\mathbf{L}^2(\mu)}, \forall i, j \in I,$$

$$(8.13) \quad \sum_{i \in I} S_i S_i^* = \text{Id}_{\mathbf{L}^2(\mu)}.$$

**Proof:** Condition (8.12) is immediate from the preceding lemma. Now fix  $i \in I$ . one checks that the  $\mathbf{L}^2(\mu)$ -adjoint of the operator in (8.11) is

$$(8.14) \quad S_i^* \varphi = \varphi \circ \tau_i, \quad \forall \varphi \in \mathbf{L}^2(\mu), \quad \forall i \in I.$$

We are now ready to verify (8.13), i.e. the second Cuntz relation. In this computation we make use of (8.7), i.e.

$$\sum_{i \in I} g_i = 1, \quad \mu \text{ a.e.}$$

For  $\varphi \in \mathbf{L}^2(\mu)$ , we have:

$$\begin{aligned} \int_M |\varphi(x)|^2 d\mu(x) &= \sum_{i \in I} \int_M |\varphi(x)|^2 g_i(x) d\mu(x) \\ &= \sum_{i \in I} \int_M |\varphi(x)|^2 d(\mu \circ \tau_i^{-1})(x) \\ &= \sum_{i \in I} \int_M |\varphi \circ \tau_i|^2(x) d\mu(x) \\ &= \sum_{i \in I} \int_M |S_i^* \varphi|^2(x) d\mu(x) \\ &= \sum_{i \in I} \langle \varphi, S_i S_i^* \varphi \rangle_{\mathbf{L}^2(\mu)}. \end{aligned}$$

Since this holds for all  $\varphi \in \mathbf{L}^2(\mu)$  the desired formula (8.13) has been verified.  $\square$

## 9. GAUSSIAN VERSUS NON-GAUSSIAN

In this section we show that the theory, developed above, initially for Gaussian Hilbert spaces, applies to some non-Gaussian cases; for example to those arising in the study of random functions. To make this point specific, we address such a problem for the special case of a concrete random power series, studied as a family of infinite Bernoulli convolutions on the real line. We know, see [40], that every positive definite function may be realized in a Gaussian Hilbert space. Our results in Sections 4-3 are making this precise in some settings dictated

by applications to stochastic integration.

**Definition 9.1.** *If  $T$  is a set, then the function*

$$(9.1) \quad C : T \times T \longrightarrow \mathbb{C}$$

*is said to be positive semi-definite (p.s.d) (we will also say positive definite) if for every finite subset  $S \subset T$ , and every family  $\{a_s\}_{s \in S} \subset \mathbb{C}^{|S|}$ , we have*

$$(9.2) \quad \sum_{(s,t) \in S \times S} \overline{a_s} a_t C(s,t) \geq 0.$$

A Gaussian representation of a p.s.d function consists of a Hilbert space  $\mathcal{H}$  and a function

$$X : T \longrightarrow \mathcal{H}$$

such that

$$(9.3) \quad C(s,t) = \langle X_s, X_t \rangle_{\mathcal{H}}, \quad \forall t, s \in T,$$

such that, for all  $t \in T$ ,  $X_t$  is a Gaussian random variable with zero mean,  $\mathbb{E}(X_t) = 0$ , and moreover

$$(9.4) \quad \mathbb{E}(X_s^* X_t) = C(s,t).$$

The following is an important example of a solution to the problem (9.2)–(9.4), when the Gaussian restriction is relaxed. In its simplest form, it may be presented as follows:

**Proposition 9.2.** *Let  $T = (0, 1)$  and consider the function*

$$C : (0, 1) \times (0, 1) \longrightarrow \mathbb{R}^+$$

*defined by*

$$(9.5) \quad C(\lambda, \rho) = \frac{\lambda \rho}{1 - \lambda \rho}.$$

*There is a solution to the representation problem (9.3) in a binary probability space  $\Omega(2) = \times_{\mathbb{N}} \{\pm 1\}$  with the infinite coin-tossing probability product measure*

$$q := \times_{\mathbb{N}} \left( \frac{1}{2}, \frac{1}{2} \right).$$

**Proof:** We will be making use of facts on Bernoulli convolutions. For some of the fundamentals in the theory of Bernoulli convolutions, we refer to [29, 41, 42]. We consider on  $\Omega(2)$  the system  $\{\epsilon_k\}_{k \in \mathbb{N}}$  of random variables

$$\epsilon_k((\omega_j)_{j \in \mathbb{N}}) = \omega_k, \quad \forall k \in \mathbb{N}.$$



Denoting the expectation with respect to  $q$  by  $\mathbb{E}_q(\cdot)$  we have

$$(9.6) \quad \mathbb{E}_q(\epsilon_k) = 0, \quad \text{and} \quad \mathbb{E}_q(\epsilon_j \epsilon_k) = \delta_{j,k}, \quad \forall j, k \in \mathbb{N}.$$

The system  $\{\epsilon_k\}_{k \in \mathbb{N}}$  is therefore i.i.d., but non-Gaussian. For  $\lambda \in (0, 1)$ , set

$$(9.7) \quad X_\lambda(\omega) = \sum_{k=1}^{\infty} \epsilon_k(\omega) \lambda^k, \quad \forall \omega \in \Omega(2).$$

Such an expression is called a random power series. Then the distribution

$$(9.8) \quad \mu_\lambda = q \circ X_\lambda^{-1},$$

(i.e.  $\mu_\lambda(A) = q(X_\lambda^{-1}(A))$  for all Borel subsets  $A$  of  $\Omega(2)$ ) is the infinite Bernoulli convolution measure given by its Fourier transform

$$(9.9) \quad \widehat{\mu}(\xi) = \prod_{n=1}^{\infty} \cos(2\pi \lambda^n \xi), \quad \forall \xi \in \mathbb{R}.$$

Equivalently, if  $\tau_\pm(x) = \lambda(x \pm 1)$ , the  $\mu_\lambda$  is the unique measure defined on the Borel sigma-algebra  $\mathcal{B}$  of  $\mathbb{R}$  by

$$(9.10) \quad \mu_\lambda = \frac{1}{2} (\mu_\lambda \circ \tau_+^{-1} + \mu_\lambda \circ \tau_-^{-1}),$$

see also (8.8). Note that for every  $\lambda \in (0, 1)$ ,  $\mu_\lambda$  has compact support strictly contained in the open interval  $(-1, 1)$ . We now verify the covariance property

$$(9.11) \quad \mathbf{E}_q(X_\lambda X_\rho) = \frac{\lambda \rho}{1 - \lambda \rho}, \quad \forall \lambda, \rho \in (0, 1).$$

In the left hand-side of (9.11) we substitute (9.7), and we make use of the i.i.d. properties (9.6). Then

$$\mathbf{E}_q(X_\lambda X_\rho) = \sum_{k=1}^{\infty} \lambda^k \rho^k = \frac{\lambda \rho}{1 - \lambda \rho}.$$

□

**Theorem 9.3.** (*Peres-Schlag-Solomyak and Peres-Solomyak, [41, 42]*)  
There is a Borel function

$$D : \left[\frac{1}{2}, 1\right) \times (-1, 1) \longrightarrow \mathbb{R}^+$$

such that the following properties hold for all  $f \in C_c \left( \left[ \frac{1}{2}, 1 \right) \times (-1, 1) \right)$ :

(i) The integral

$$\iint_{\left[ \frac{1}{2}, 1 \right) \times (-1, 1)} f(\lambda, x) d\mu_\lambda(x) d\lambda$$

is well defined, where  $d\lambda$  denotes the standard Lebesgue measure restricted to  $\left[ \frac{1}{2}, 1 \right)$ ,

and

(ii) it holds that

$$\iint_{\left[ \frac{1}{2}, 1 \right) \times (-1, 1)} f(\lambda, x) d\mu_\lambda(x) d\lambda = \iint_{\left[ \frac{1}{2}, 1 \right) \times (-1, 1)} f(\lambda, x) D(\lambda, x) dx d\lambda.$$

We first present some corollaries of this result.

**Definition 9.4.** Set

(9.12)

$$\text{AC}_2 = \left\{ \lambda \in \left[ \frac{1}{2}, 1 \right) \mid \text{the Radon-Nikodym derivative } \frac{d\mu_\lambda}{dx} \in \mathbf{L}^2(dx) \right\}.$$

(Note that the existence is part of the definition).

**Remark 9.5.** The theorem asserts that  $\text{AC}_2$  has Lebesgue measure equal to  $1/2$ , i.e.  $\mu_\lambda$  is singular only on a subset of  $\left[ \frac{1}{2}, 1 \right)$  of measure zero. By a result of Erdős (see [18]), when  $\lambda = g^{-1}$  where  $g = \frac{\sqrt{5}+1}{2}$  is the Golden ratio, then  $\mu_\lambda$  is singular. Otherwise it is absolutely continuous on a subset in  $\left[ \frac{1}{2}, 1 \right)$  of full measure.

**Corollary 9.6.** For a number  $\lambda \in \left[ \frac{1}{2}, 1 \right)$ , the following conditions are equivalent:

(i) The function

$$t \mapsto \prod_{n=1}^{\infty} \cos(\lambda^n t), \quad t \in \mathbb{R}$$

belongs to  $\mathbf{L}^2(\mathbb{R}, dx)$ .

(ii) We have

$$\liminf_{r \downarrow 0} \frac{1}{2r} \mu_\lambda([x-r, x+x]) < \infty$$

for a.a.  $x \in \mathbb{R}$ . In this case, we may take

$$(9.13) \quad D(\lambda, x) = \liminf_{r \downarrow 0} \frac{1}{2r} \mu_\lambda([x-r, x+x]) \in \mathbf{L}^2((-1, 1), dx)$$

in (9.12).

**Corollary 9.7.** *The points  $\lambda \in \text{AC}_2$  correspond to a single equivalence class in the Hilbert space  $\mathcal{H}$  of Definition 4.1. If  $\lambda_1$  and  $\lambda_2$  belong to  $\text{AC}_2$ , we have*

$$(9.14) \quad \langle f_1 \sqrt{d\mu_{\lambda_1}}, f_2 \sqrt{d\mu_{\lambda_2}} \rangle_{\mathcal{H}} = \int_{-1}^1 f_1(x) f_2(x) \sqrt{D(\lambda_1, x) D(\lambda_2, x)} dx,$$

where  $dx$  is the Lebesgue measure.

**Proof:** Using (9.10)

$$(9.15) \quad \int \varphi(x) d\mu_{\lambda}(x) = \frac{1}{2} \left( \int \varphi(\lambda(x+1)) d\mu_{\lambda}(x) + \int \varphi(\lambda(x-1)) d\mu_{\lambda}(x) \right),$$

and a recursive iteration leads to the representation

$$(9.16) \quad \begin{aligned} \widehat{\mu_{\lambda}}(t) &= \int_{\mathbb{R}} e^{-itx} d\mu_{\lambda}(x) \\ &= \mathbb{E}_q(e^{-itX_{\lambda}}) \\ &= \prod_{n=1}^{\infty} \cos(\lambda^n t), \end{aligned}$$

with the right hand-side of (9.16) converging point-wise for all  $t \in \mathbb{R}$ .

If  $\lambda \in \text{AC}_2$ , then

$$D(\lambda, x) = \frac{d\mu_{\lambda}}{dx} \in \mathbf{L}^2(-1, 1) \subset \mathbf{L}^2(\mathbb{R}),$$

and substitution into (9.16) yields

$$\widehat{\mu_{\lambda}}(t) = \int_{\mathbb{R}} e^{-itx} D(\lambda, x) dx,$$

and by the  $\mathbf{L}^2(\mathbb{R}, dx)$ -Fourier inversion,

$$D(\lambda, x) = \int_{\mathbb{R}} e^{itx} \prod_{n=1}^{\infty} \cos(\lambda^n t) dt$$

for a.a.  $x \in (-1, 1)$ . Hence, Plancherel's equality leads to

$$\int_{-1}^1 |D(\lambda, x)|^2 dx = \int_{\mathbb{R}} \prod_{n=1}^{\infty} \cos^2(\lambda^n t) dt < \infty.$$

We now turn to (9.14). If  $\lambda_1, \lambda_2 \in \text{AC}_2$ , then both  $\mu_{\lambda_1}$  and  $\mu_{\lambda_2}$  are absolutely continuous with respect to Lebesgue measure, and by (4.3)

we get

$$\begin{aligned}\langle f_1 \sqrt{d\mu_{\lambda_1}}, f_2 \sqrt{d\mu_{\lambda_2}} \rangle_{\mathcal{H}} &= \int_{-1}^1 f_1(x) f_2(x) \sqrt{\frac{d\mu_{\lambda_1}}{dx}(x) \frac{d\mu_{\lambda_2}}{dx}(x)} dx \\ &= \int_{-1}^1 f_1(x) f_2(x) \sqrt{D(\lambda_1, x) D(\lambda_2, x)} dx.\end{aligned}$$

□

We showed that, when  $\lambda$  is given in  $\text{AC}_2$ , then the corresponding Bernoulli measure  $\mu_\lambda$  satisfies the Bernoulli scaling law. But for  $\lambda$  fixed in  $\text{AC}_2$ , this then turns into a scaling identity for the  $\mathbf{L}^2$  Radon-Nikodym derivative, a variant of the scaling law studied in wavelet theory, but so far only for rational values of  $\lambda$ . This fact is isolated in the corollary below. It is of interest since there is very little known about  $\mathbf{L}^2$  solutions to scaling identity for non-rational values of  $\lambda$ . For the literature on this we cite [13, 14, 15, 50].

**Corollary 9.8.** *Let  $(\mu_\lambda)_{\lambda \in (0,1)}$  be the Bernoulli measures. For  $\lambda \in \text{AC}_2$ , let  $D(\lambda, \cdot) = \frac{d\mu_\lambda(x)}{dx}$  be the Radon-Nikodym derivative. Extend  $D(\lambda, x)$  to  $x \in \mathbb{R}$  by setting it to be equal to zero in the complement of  $(-1, 1)$ . Then,*

$$D_\lambda(\cdot) = D(\lambda, \cdot) \in \mathbf{L}_+^1(\mathbb{R}), \quad \int_{\mathbb{R}} D(\lambda, x) dx = 1,$$

and

$$(9.17) \quad D_\lambda(\lambda x) = \frac{1}{2} (D_\lambda(x+1) + D_\lambda(x-1))$$

for a.a.  $x$  with respect to the Lebesgue measure on  $\mathbb{R}$ .

**Proof:** From the definition of  $\text{AC}_2$  we know that the Radon-Nikodym derivative  $x \mapsto D(\lambda, x)$  exists, and that  $D(\lambda, \cdot) \in \mathbf{L}_+^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ . Using (9.15)-(9.16) above, we conclude that  $\int_{\mathbb{R}} D(\lambda, x) dx = 1$ . □

**Remark 9.9.** *Note that for  $\lambda = \frac{1}{2}$ , equation (9.17) reduces to the standard scaling identity for the Haar wavelet system in  $\mathbf{L}^2(\mathbb{R}, dx)$ . In wavelet theory, the scaling identity is considered for  $N \in \mathbb{Z}_+$ ,  $N > 1$ , as follows: Given  $N$ , one studies solutions  $\varphi \in \mathbf{L}^2(\mathbb{R}, dx)$  to the scaling-rule*

$$\varphi\left(\frac{x}{N}\right) = \sum_{k \in \mathbb{Z}} a_k \varphi(x - k), \quad \text{a.a. } x,$$

where  $(a_k)_{k \in \mathbb{Z}}$  is square summable.

Before giving the proof of Theorem 9.3 we need preliminary lemmas:

**Lemma 9.10.**

- (i) If  $\lambda \in (0, \frac{1}{2})$ , the measure  $\mu_\lambda$  is singular with respect to Lebesgue measure, with scaling dimension  $D_s = -\frac{\ln 2}{\ln \lambda}$ , and the IFS defined by  $x \mapsto \lambda(x \pm 1)$  is "non-overlapping".
- (ii) If  $\lambda = \frac{1}{2}$ , then  $\mu_\lambda$  is equal to the Lebesgue measure restricted to  $[-1, 1]$ .
- (iii) For almost all  $\lambda$  in  $[\frac{1}{2}, 1)$ , the measure  $d\mu_\lambda$  is absolutely continuous with respect to  $dx$ , with Radon-Nikodym derivative

$$\frac{d\mu_\lambda}{dx}(x) = D(\lambda, x) \in \mathbf{L}^2(-1, 1).$$

**Proof:** The first two assertions are in the literature, and (iii) is from [42]. It is our aim in Theorem 9.3 to give an independent proof in the reproducing kernel (9.5) restricted to  $[\frac{1}{2}, 1) \times [\frac{1}{2}, 1)$ ; see also Proposition 9.2 and equation (9.10).  $\square$

Our purpose in connection with Theorem 9.3 is as follows: The proof of the result in [42] relies on the following estimate on  $X_\lambda$  for a subset of points  $\lambda \in (\frac{1}{2}, 1)$ , defined for measurable functions  $F$  on  $(\times_{\mathbb{N}} \{\pm 1\}) \times (\times_{\mathbb{N}} \{\pm 1\})$ , estimating expectations

$$(9.18) \quad \mathbb{E}_{q \times q}((\mathbf{1} \otimes X_\lambda - X_\lambda \otimes \mathbf{1}) F)$$

where  $\mathbf{1}$  is the constant function 1 on  $\times_{\mathbb{N}} \{\pm 1\}$ .

One is in particular interested in (9.18) in functions  $F$  of the form

$$(9.19) \quad F_r = \chi_{\{(\omega, \omega') \text{ such that } |X_\lambda(\omega) - X_\lambda(\omega')| \leq r\}},$$

where  $r \geq 0$ .

For subintervals  $J$  of  $(\frac{1}{2}, 1)$  one must find estimate on

$$\int_J \mathbb{E}_{q \times q}(F_r) d\lambda$$

In accomplishing this, the following three lemmas below are helpful.

**Lemma 9.11.** Let  $\mathcal{H}$  be the reproducing kernel Hilbert space from (9.5), with  $\lambda, \rho \in [\frac{1}{2}, 1)$ , and set

$$k_\lambda(\rho) = \frac{\lambda\rho}{1 - \lambda\rho} = \langle k_\lambda, k_\rho \rangle_{\mathcal{H}}, \quad \forall \lambda, \rho \in [\frac{1}{2}, 1).$$

Then the assignment

$$(9.20) \quad k_\lambda \in \mathcal{K} \quad \mapsto \quad X_\lambda(\cdot) \in \mathbf{L}^2(\times_{\mathbb{N}} \{\pm 1\}, q)$$

extends to a Hilbert space isometry of  $\mathcal{H}$  into  $\mathbf{L}^2(\times_{\mathbb{N}} \{\pm 1\}, q)$ .

**Proof:** The conclusion follows from the basic axioms of reproducing kernel Hilbert spaces once we verify that

$$(9.21) \quad \langle k_\lambda, k_\rho \rangle_{\mathcal{H}} = \int_{\times_{\mathbb{N}} \{\pm 1\}} X_\lambda(\omega) X_\rho(\omega) dq(\omega),$$

equation (9.3) from the computation

$$\langle k_\lambda, k_\rho \rangle_{\mathcal{H}} = \frac{\lambda \rho}{1 - \lambda \rho} = \mathbb{E}_q(X_\lambda X_\rho),$$

by (9.5). □

**Definition 9.12.** We denote by  $\mathbf{H}^2(\mathbb{D})$  the Hardy space of the open unit disk of functions analytic in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$  represented as

$$(9.22) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

and norm  $\|f\|_{\mathbf{H}^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |a_n|^2$ , and set

$$\mathbf{H}_0^2(\mathbb{D}) = \{f \in \mathbf{H}^2(\mathbb{D}) \mid f(0) = 0\}.$$

**Lemma 9.13.** The reproducing kernel Hilbert space  $\mathcal{H}$  from (9.5) is isometrically equal to  $\mathbf{H}_0^2(\mathbb{D})$  via the map

$$(9.23) \quad k_\lambda \in \mathcal{H} \quad \mapsto \quad \tilde{k}_\lambda(z) = \sum_{n=1}^{\infty} \lambda^n z^n \in \mathbf{H}_0^2(\mathbb{D}).$$

**Proof:** It is immediate from the definition that the map  $k_\lambda \mapsto \tilde{k}_\lambda$  in (9.23) extends to an isometry

$$J : \mathcal{H} \longrightarrow \mathbf{H}_0^2(\mathbb{D}).$$

We claim that it is onto:  $\text{ran}(J) = \mathbf{H}_0^2(\mathbb{D})$ . Indeed, since  $J$  is isometric,  $\text{ran } J$  is closed. now, if  $f \in \mathbf{H}_0^2(\mathbb{D}) \ominus \text{ran } J$ , then

$$f(\lambda) = \langle f, \tilde{k}_\lambda \rangle_{\mathbf{H}_0^2(\mathbb{D})} = 0, \forall \lambda \in [\frac{1}{2}, 1).$$

Since  $f$  is analytic in  $\mathbb{D}$  and  $[\frac{1}{2}, 1) \subset \mathbb{D}$ , we conclude that  $f \equiv 0$ , and therefore  $\text{ran}(J) = \mathbf{H}_0^2(\mathbb{D})$  as claimed. □

We now comment on the use of Lemmas 9.11 and 9.13. About (9.18) the estimate

$$|\mathbb{E}_{q \times q}((\mathbf{1} \otimes X_\lambda - X_\lambda \otimes \mathbf{1}) F)| \leq \sqrt{\frac{2\lambda^2}{1-\lambda^2}} \|F\|_{\mathbf{L}^2(q \times q)}$$

follows from the Cauchy-Schwarz inequality, using that

$$\mathbf{1} \otimes X_\lambda \perp X_\lambda \otimes \mathbf{1}$$

in  $\mathbf{L}^2(q \times q)$ , and

$$\|\mathbf{1} \otimes X_\lambda\|_{\mathbf{L}^2(q \times q)}^2 = \|X_\lambda\|_{\mathbf{L}^2(q)}^2 = \frac{\lambda^2}{1-\lambda^2}.$$

See Proposition 9.2 and Lemma 9.11.

As for estimating (9.19), we make use of the Hardy space representation in Lemma 9.13. Under the isometry in (9.21) the difference  $|X_\lambda(\omega) - X_\lambda(\omega')|$  with  $\omega_i = \omega'_i$  for  $i = 1, 2, \dots, k$  may be estimated in the subspace  $z^k \mathbf{H}_0^2(\mathbb{D})$ , i.e. functions in  $\mathbf{H}^2(\mathbb{D})$  vanishing at 0 to order  $k+1$ .

## 10. BOUNDARIES OF POSITIVE DEFINITE FUNCTIONS

In this section we apply our results from Sections 3 and 7 into a general boundary analysis for an arbitrarily given non-degenerate positive definite function (Definition 9.1). While it is known that every non-degenerate positive definite function admits a Gaussian representation, our construction here offers such a representation in a form of a boundary in a sense which naturally generalizes boundaries in classical analysis, for example generalizing the known boundary analysis for the Szegő kernel of the disk. Again we stress that our starting point now is an arbitrary fixed non-degenerate positive definite function  $C$ , but  $C$  is on  $T \times T$  where  $T$  may be any set, continuous or discrete. For example  $T$  may represent the vertices in some infinite graph, and  $C$  may be some associated energy form of the graph  $G$ , induced by an electric network of  $G$ ; see e.g., [17, 31]. A second recent application of reproducing kernels and their RKHSs, is the theory of (supervised) learning; see e.g., [35, 39, 48]. The problem there is a prediction of outputs based on observed samples; and for this the kernel enters in representations of samples.

Among the applications of stochastic processes, the theory of “boundaries” is noteworthy. Common to these is the need for representations

of functions on some set, say  $T$ , as integrals over some measure boundary space arising as a limiting operation derived from the points in the initial set  $T$ . As example of this is the Hardy space  $\mathbf{H}^2(\mathbb{D})$  (see Definition 9.12), which is the reproducing kernel Hilbert space with kernel the Szegő kernel

$$(10.1) \quad C(z, w) = \frac{1}{1 - zw^*}, \quad z, w \in \mathbb{D}.$$

If  $\langle \cdot, \cdot \rangle_{\mathbf{H}^2(\mathbb{D})}$  is the inner product of  $\mathbf{H}^2(\mathbb{D})$  we have

$$(10.2) \quad f(w) = \langle f, C_w \rangle_{\mathbf{H}^2(\mathbb{D})}, \quad f \in \mathbf{H}^2(\mathbb{D}), \quad w \in \mathbb{D}.$$

In this example we have

$$(10.3) \quad \frac{1}{1 - zw^*} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - ze^{-i\theta}} \frac{1}{1 - w^* e^{i\theta}} d\theta.$$

Now recall that the general case of positive definite functions  $C$  on an arbitrary set  $T$ , as in Definition 9.1, offers a generalization of the classical theory of the Hardy space recalled above. In this general case, the aim is to provide a Gaussian measure space associated to an arbitrary given positive definite function

$$(10.4) \quad C : T \times T \longrightarrow \mathbb{C}.$$

This measure space will be denoted by  $\text{bdr}_C(T)$ , and it should be offer a direct integral representation for (10.4) naturally generalizing (10.3), where the boundary of  $\mathbb{D}$  from (10.3) is the circle  $\{z \in \mathbb{C} \mid |z| = 1\}$ .

**Definition 10.1.** *We say that a positive definite function  $C$  on a set  $T$  is non-degenerate if the following two conditions are satisfied:*

(i)

$$\dim \mathcal{H}(C) = \aleph_0,$$

where  $\mathcal{H}(C)$  is the reproducing kernel Hilbert space associated to  $C$ .

(ii) *The following implication holds:*

$$C(s, t_1) = C(s, t_2), \quad \forall s \in T \implies t_1 = t_2.$$

**Theorem 10.2.** *Let  $C : T \times T \longrightarrow \mathbb{C}$  be a non-degenerate positive definite function where  $T$  is some fixed set. Let  $\mathbf{s}'$  be the sequence space introduced in Lemma 3.1 (see equation (3.2)). Then there is a weak\*-closed subspace  $\text{bdr}_C(T) \subset \mathbf{s}'$ , a Gaussian measure  $\mathbb{P}_C$  defined on the cylinder sigma-algebra in  $\text{bdr}_C(T)$ , and a Gaussian process  $X$ :*

$$(10.5) \quad X_t : \text{bdr}_C(T) \longrightarrow \mathbb{C}, \quad t \in T,$$

such that (i) we have

$$(10.6) \quad C(s, t) = \int_{\text{bdr}_C(T)} X_s(\xi)^* X_t(\xi) d\mathbb{P}_C(\xi), \quad \forall s, t \in T,$$



and, (ii)  $(\text{bdr}_C(T), \mathbb{P}_C, X_t)$  is a minimal solution to (i).

**Proof:** Let  $\{\varphi_j\}_{j \in \mathbb{N}}$  be an orthonormal basis in  $\mathcal{H}(C)$ . It is well known that

$$(10.7) \quad C(t, s) = \sum_{j=1}^{\infty} \varphi_j(t) \varphi_j(s)^*, \quad \forall t, s \in T,$$

and

$$(10.8) \quad \sum_{j=1}^{\infty} |\varphi_j(t)|^2 = C(t, t) < \infty.$$

Now define  $\tau : T \longrightarrow \ell^2 \subsetneq \mathbf{s}'$  by

$$(10.9) \quad \tau(t) = (\varphi_j(t))_{j \in \mathbb{N}}, \quad t \in T.$$

We claim that  $\tau$  is one-to-one, and as result, we may identify points  $t \in T$  with their image in  $\mathbf{s}'$ . Indeed, let  $t_1, t_2 \in T$  and suppose that  $\tau(t_1) = \tau(t_2)$ . Then,

$$C(t, t_1) = \sum_{j=1}^{\infty} \varphi_j(t) (\varphi_j(t_1))^* = \sum_{j=1}^{\infty} \varphi_j(t) (\varphi_j(t_2))^* = C(t, t_2),$$

and in view of condition (ii) in Definition 10.1 we conclude that  $t_1 = t_2$ .

Set  $\tau(T) = \{\tau(t) \mid t \in T\}$ , and set  $\text{clo}_C(T)$  its closure in  $\mathbf{s}'$ . Here, by closure we mean the weak\*-topology in  $\mathbf{s}'$  defined by the duality between  $\mathbf{s}$  and  $\mathbf{s}'$ . The neighborhoods for this topology are generated by the cylinder sets introduced in (3.3). Finally, set

$$(10.10) \quad \text{bdr}_C(T) = \text{clo}_C(T) \setminus \tau(T).$$

Now, following Lemma 3.1, set for  $\xi \in \text{bdr}_C(T)$

$$(10.11) \quad X_t(\xi) = \sum_{j=1}^{\infty} (\varphi_j(t))^* \pi_j(\xi) = \sum_{j=1}^{\infty} \xi_j (\varphi_j(t))^*,$$

the “random” function associated with the choice  $\{\varphi_j\}$  of ONB in  $\mathcal{H}(C)$ . Note that if  $\xi$  in (10.11) is “deterministic”, i.e., if there is a  $s \in T$  such that

$$\pi_j(\xi) = \xi_j = \varphi_j(s), \quad \forall j \in \mathbb{N},$$

then

$$(10.12) \quad X_t(\xi) = \sum_{j=1}^{\infty} \varphi_j(s) (\varphi_j(t))^* = C(t, s), \quad \forall t \in T.$$

Now, define by  $\mathbb{P}_C$  the measure on  $\text{bdr}_C(T)$  induced by  $Q$  on  $\mathbf{s}'$ , as in Theorem 7.1. We get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_C}(X_t(\cdot)X_s(\cdot)^*) &= \mathbb{E}_{\mathbb{P}_C}\left(\left(\sum_{j=1}^{\infty}\varphi_j(s)\pi_j^*\right)\left(\sum_{k=1}^{\infty}\varphi_k(t)^*\pi_k\right)\right) \\ &= \sum_{j=1}^{\infty}\varphi_j(s)\varphi_j(t)^* \\ &= C(t, s), \quad \forall t, s \in T,\end{aligned}$$

whence the desired conclusion (10.6) in part (i) of the theorem. The other conclusion (ii) follows from the assignment (10.10) in the definition of  $\text{bdr}_C(T)$ .  $\square$

**Application 10.3.** *Our boundary construction applies to electrical networks as follows (see [30]).*

*An electrical network is an infinite graph  $(V, E, c)$ ,  $V$  for vertices, and  $E$  for edges, where  $c$  is a positive function on  $E$ , representing conductance. As sketched in [30], we get a reproducing kernel Hilbert space from the energy form of  $(V, E, c)$ . In [30], the authors propose one boundary construction, and one can verify that the one from our present Theorem 10.2 applied to  $\mathcal{H}$  is a refinement.*

**Remark 10.4.** *Our construction of  $\text{bdr}_C(T)$  depends on the choice of ONB in (10.7), but the arguments in the proof in Theorem 10.2 above) show that two choices of ONB  $\{\varphi_j\}_{j \in \mathbb{N}}$  and  $\{\psi_k\}_{k \in \mathbb{N}}$  yields the same  $\text{bdr}_C(T)$  if and only if there is an infinite unitary matrix  $(U_{j,k})_{(j,k) \in \mathbb{N}^2}$  such that*

$$(i) \quad \varphi_j = \sum_{k \in \mathbb{N}} U_{j,k} \psi_k,$$

*and the following equivalence holds:*

$$(ii) \quad (b_j)_{j \in \mathbb{N}} \in \mathbf{s} \iff (c_j)_{j \in \mathbb{N}} \in \mathbf{s}, \text{ with } c_j = \sum_{k \in \mathbb{N}} U_{j,k} b_k.$$

*In other words, the matrix-operation defined from  $U$  preserves the sequence space  $\mathbf{s}$  of (3.1).*

**Example 10.5.** *Consider now the Hardy space  $\mathbf{H}^2(\mathbb{D})$  (see Definition 9.12). One may check that, with the choice of the standard ONB in  $\mathbf{H}^2(\mathbb{D})$*

$$\varphi_k(z) = z^k, \quad k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad z \in \mathbb{D},$$

we get

$$\text{bdr}_{\text{Szegö}} = \left\{ (e^{ik\theta})_{k \in \mathbb{N}_0} \mid \theta \in (-\pi, \pi] \right\},$$

which by identification yields  $(\pi, \pi]$ , which is consistent with (10.3) above.

**Example 10.6.** (see [4]). Here the pair  $(C, T)$  from Definition 10.1 is as follows: Consider the rational function  $R(z)$  given by

$$R(z) = z^4 - 2z^2, \quad z \in \mathbb{C}.$$

Set  $R_0(z) = z$ ,  $R_1(z) = R(z)$  and

$$R_n(z) = \underbrace{(R \circ R \circ \cdots \circ R)}_{n \text{ times}}(z).$$

Now set

$$T = \Omega = \{z \in \mathbb{C} \text{ such that } (R_n(z))_{n \in \mathbb{N}_0} \in \ell^1\},$$

(where  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ) and on  $\Omega \times \Omega$  set

$$C(z, w) = \prod_{n=0}^{\infty} (1 + R_n(z)R_n(w)^*).$$

Using the ideas of Exemple 10.5 and from [4] we note that for this  $(C, T)$  we get that  $\text{clo}_C(T)$  is the filled Julia set of  $R$ . See also citeMR1128089 for basic properties of Julia sets derived from fixed rational functions of a single complex variable.

**Definition 10.7.** Let  $(C, T)$  be as in Definition 10.1. Following [48], we say that  $C$  is a Mercer-kernel if:

- (i)  $T$  is a compact metric space (with respect to some metric, say  $d$ ), and
- (ii) The function

$$C : T \times T \longrightarrow \mathbb{C}$$

is continuous with respect to  $d \times d$ .

**Proposition 10.8.** If  $(C, T)$  is a Mercer kernel, then  $\text{clo}_C(T) = \tau(T)$ ; in other words  $\tau(T)$  from (10.9) is closed.

**Proof:** Let  $\xi \in \mathbf{s}'$ , and let  $(t_k)_{k \in \mathbb{N}}$  be a sequence of points of  $T$  such that  $\lim_{k \rightarrow \infty} \tau(t_k) = \xi$ ; see the discussion before Lemma 3.1. Using (i) in Definition 10.7, we may, without loss of generality, assume that the sequence  $(t_k)_{k \in \mathbb{N}}$  is convergent in  $T$ , i.e.  $\lim_{k \rightarrow \infty} d(t, t_k) = 0$  where  $t \in T$  is its limit point.

Let  $\{\varphi_k\}_{k \in \mathbb{N}}$  be an ONB in  $\mathcal{H}(C)$ , see (10.7) in the proof of Theorem 10.2. Then,

$$\begin{aligned}\|\tau(t) - \tau(t_k)\|_2^2 &= \sum_{j=1}^{\infty} |\varphi_j(t) - \varphi_j(t_k)|^2 \\ &= C(t, t) - 2\operatorname{Re} C(t_k, t) + C(t_k, t_k),\end{aligned}$$

where we have used (10.7)-(10.9) in this computation.

By virtue of Condition (ii) in Definition 10.7, we now note that the right hand-side in the last term converges to zero as  $k \rightarrow \infty$ . But convergence in  $\ell^2$  of the sequence  $(\tau(t_k))_{k \in \mathbb{N}}$  implies convergence in  $\mathbf{s}'$ . We conclude that  $\tau(t) = \xi$ , and so  $\tau(T)$  is closed in  $\mathbf{s}'$ .  $\square$

**Example 10.9.** Let  $T = I = [0, 1]$  be the closed unit interval, and set

$$C(t, s) = t \wedge s, \quad t, s \in I.$$

Set

$$(10.13) \quad \varphi_k(t) : \begin{cases} \sqrt{2} \frac{\sin k\pi t}{k\pi}, & k \in \mathbb{N}, \\ t, & k = 0. \end{cases}$$

Then:

(i)  $\tau(t) := (\varphi_k(t))_{k \in \mathbb{N}_0}$  satisfies

$$(10.14) \quad \|\tau(t) - \tau(s)\|_2^2 = |t - s|, \quad t, s \in [0, 1].$$

(ii) The map  $t \mapsto \tau(t)$  is an homeomorphism from  $I$  onto a closed curve starting at  $v_0 = (0, 0, 0, \dots)$  and with endpoint  $v_1 = (1, 0, 0, \dots)$  in  $\ell^2$ .

(iii) The curve in (ii) has no self-intersection.

**Proof of the claims in Example 10.9:** The conclusions are immediate from Proposition 10.8. Indeed, the reproducing kernel Hilbert space associated to  $C$  is

$$(10.15) \quad \mathcal{H} = \{f \in \mathbf{L}^2(I) \mid f' \in \mathbf{L}^2(I) \text{ and } f(0) = 0\},$$

and one easily checks that the function system  $(\varphi_k)_{k \in \mathbb{N}_0}$  is an orthonormal basis in  $\mathcal{H}$ . Indeed, for  $j, k \in \mathbb{N}$ ,

$$\langle \varphi_j, \varphi_k \rangle_{\mathcal{H}} = 2 \int_0^1 \cos(j\pi x) \cos(k\pi x) dx = \delta_{j,k}.$$

The assertions follow then from Proposition 10.8. In this example the Gaussian process from (10.11) associated with  $(C, I)$  is the Brownian

motion. Hence

$$(10.16) \quad \|\tau(t) - \tau(s)\|_2^2 = \mathbb{E}(|X_t - X_s|^2) = |t - s|, \quad t, s \in I,$$

which is (i), and also leads to (ii) since  $\tau$  is one-to-one and continuous between two compact spaces, and so is an homeomorphism. To justify (10.16) note that the Hilbert norm in  $\mathcal{H}$  is  $\|f\|_{\mathcal{H}}^2 = \int_0^1 |f'(x)|^2 dx$ .

Setting

$$C_t(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x \leq t, \\ t, & t < x, \end{cases}$$

we get

$$\begin{aligned} \langle C_t, C_s \rangle_{\mathcal{H}} &= \int_0^1 \chi_{[0,t]}(x) \chi_{[0,s]}(x) dx \\ &= t \wedge s \\ &= C(t, s), \end{aligned}$$

and

$$C(t, s) = ts + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin(k\pi t) \sin(k\pi s)}{k^2}.$$

Finally, if there exist  $t_1$  and  $t_2$  in  $(0, 1)$  such that  $\tau(t_1) = \tau(t_2)$ , then  $C(t, t_1) = C(t, t_2)$  which is not possible unless  $t_1 = t_2$ .  $\square$

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